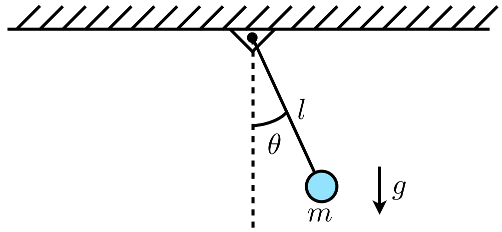

Exercise 3–1: Consider the schematic of the single pendulum.



The kinetic energy T and potential energy V may be written as:

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$
$$V = -gml\cos(\theta)$$

The Lagrangian $\mathcal{L}$ is given by $\mathcal{L} = T - V$, and the Euler-Lagrange equations for the motion of the pendulum are given by the following second order differential equation in θ:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

Write down the second order ODE using the specific T and V defined above. Please write this ODE in the form $\ddot{\theta} = f(\theta, \dot{\theta})$. Notice that this ODE is not linear!!

Now you may assume that $l = m = g = 1$ for the remainder of the problem.

You may still suspend variables to get a system of two first order (nonlinear) ODEs by writing the ODE as:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = f(\theta, \omega)$$

(a) What are the fixed points of this system where all derivatives are zero?

(b) Write down the linearized equations in a neighborhood of each fixed point and determine the linear stability. You may formally linearize the nonlinear ODE or you may use a small angle approximation for $\sin(\theta)$; the two approaches are equivalent.

(c) Compute the eigenvalues and eigenvectors for each fixed point.

(d) Sketch the linearized phase portrait in a small neighborhood around each of these fixed points.

(e) How do the above answers match your physical intuition about the fixed points of this system?

Exercise 3–2: Consider the following ODE with very nearly parallel eigenvectors:

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -0.01 & 0 \\ 1 & -0.011 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

(a) Is the system stable? Please explain why or why not (note that you can do any calculations either by hand or in MATLAB, as long as you explain what you are doing and why).

(b) Compute the solution of this ODE using MATLAB (`ode45`) for the initial condition $\begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$. Plot the trajectory in the x-v plane. Does the solution make sense given your answer to part (a)?

(c) If v is velocity, so kinetic energy is $T = \frac{1}{2}v^2$, plot the kinetic energy against time?

This example illustrates the phenomena of **transient energy growth** experienced by stable systems that have nearly parallel eigenvectors corresponding to very similar eigenvalues. These systems are called non-normal (i.e., $AA^T \neq A^T A$). This is common to many systems, and is responsible for the transition from a laminar solution to a turbulent solution in many fluid systems (because the transient energy growth takes the fluid far enough away from the stable fixed point so that nonlinearities become large and develop into turbulence).

Exercise 3–3: Please solve the following differential equation (with initial conditions) for the three cases below (by hand!). You may use whatever method you find simplest. You may check your work in MATLAB.

$$\begin{aligned} &\ddot{x} + 6\dot{x} + 8x = f(t), \\ &x(0) = 2, \\ &\dot{x}(0) = -6. \end{aligned}$$

(a) For $f(t) = 0$. Note that this is just the unforced ODE $\ddot{x} + 6\dot{x} + 8x = 0$.

(b) For $f(t) = 6e^{-t}$.

In all cases, be sure to make sure that your initial conditions are still satisfied!

Exercise 3–4: Consider the following physical system of a bead constrained to move in the potential field $\mathbb{V}(x)$. Note: there are no equations given, and so you must use physical intuition.

- Please sketch the phase portrait (position x vs. velocity v) for the Please try to make your sketch as accurate as possible using as much information about the potential as you can. You may draw the phase portrait without any damping (i.e., no friction).
- Comment briefly on how the plot will change if we turn on friction. (you do not need to sketch anything)
- How many fixed points does the system have and what is their stability?
- What can we say about the eigenvalues of the linearization around each point?