Take home Midterm

2) a) [2] Identify what kind of algebraic object $\mathbb{Z}_5[x]/\langle x^2+1\rangle$ is.

$\langle x^2+1\rangle = \{a(x^2+1) \mid a \in \mathbb{Z}_5\}$

Note $\langle x^2+1\rangle$ is a subgroup of $\mathbb{Z}_5[x]$.

sense/notation +.5
reasons +1.5

Since $\mathbb{Z}_5[x]$ is abelian $\langle x^2+1\rangle$ is normal. Thus

$\mathbb{Z}_5[x]/\langle x^2+1\rangle$ is a $\boxed{\text{Group.}}$

Factor group of an abelian group is abelian $\Rightarrow$ $\boxed{\text{Abelian Group}}$

Is $\langle x^2+1\rangle$ an ideal? note $x(x^2+1) \notin \langle x^2+1\rangle$

$\mathbb{Z}_5[x]/\langle x^2+1\rangle$ is $\boxed{\text{not a Ring}}$

b) [3] Identify what kind of algebraic object $\mathbb{Z}_5[x]/(x^2+1)$ is

$(x^2+1) = \{(x^2+1)f(x) \mid f(x) \in \mathbb{Z}_5[x]\}$ generated as an ideal

polynomials with x^2+1 as a factor

Note (x^2+1) is closed by addition: $(x^2+1)f(x) + (x^2+1)g(x) = (x^2+1)[f(x)+g(x)]$

identity: $0 \cdot (x^2+1) = 0$

additive inverse: $-[(x^2+1)f(x)]$ ✓

$\Rightarrow$ subgroup

$\mathbb{Z}_5[x]$ is abelian $\Rightarrow$ (x^2+1) is normal. Thus

(+.5)

$\mathbb{Z}_5[x]/(x^2+1)$ is a $\boxed{\text{Group}}$

reasoning (+.5)

Factor group of an abelian group is abelian $\Rightarrow$ $\boxed{\text{Abelian Group}}$

Note (x^2+1) is an ideal $\Rightarrow$ multiplicative operator is defined

(+.5)

$\Rightarrow$ $\boxed{\text{RING}}$ Factor ring of commutative ring $\Rightarrow$ $\boxed{\text{Commutative}}$

reasoning (+.5)

Note $1 \in \mathbb{Z}_5[x]$ and $[g(x)+(x^2+1)] \cdot 1 = g(x)+(x^2+1)$

$\Rightarrow$ has $\boxed{\text{Multiplicative identity/unity}}$

To decide if integral domain, consider if (x^2+1) is prime.

$(x+2)(x+3) = x^2+2x+3x+6 = x^2+5x+6 = x^2+1$ in $\mathbb{Z}_5[x]$

BA $x+2 \notin (x^2+1)$ and $x+3 \notin (x^2+1)$

(1.5) So (x^2+1) is NOT prime $\Rightarrow \mathbb{Z}_5[x]/(x^2+1)$ is $\boxed{\text{not ID}}$

reasoning (+.5) $\Rightarrow \boxed{\text{not field}}$

c) [2] Find a representative of $(3x+4)\cdot(x+1)$ in $\mathbb{Z}_5[x]/(x^2+1)$ that has degree less than 2, if possible.

$(3x+4)(x+1) = 3x^2+3x+4x+4 = 3x^2+7x+4 = 3x^2+2x+4$

in $\mathbb{Z}_5[x]/(x^2+1)$ $\quad x^2+1+(x^2+1) = (x^2+1)$

$\Rightarrow x^2+(x^2+1) = -1+(x^2+1)$

$\Rightarrow x^2+(x^2+1) = 4+(x^2+1)$

in $\mathbb{Z}_5[x]/(x^2+1)$ then $(3x+4)(x+1)+(x^2+1) = 3x^2+2x+4+(x^2+1)$

$= 3(4)+2x+4+(x^2+1)$

$= 12+2x+4+(x^2+1)$

$= 16+2x+(x^2+1)$

$= 2x+1+(x^2+1)$

OR

remainder when $(3x+4)(x+1) = 3x^2+2x+4$ is divided by x^2+1

$$\begin{array}{r} 3 \\ x^2+0x+1 \,\overline{)\, 3x^2+2x+4} \\ -(3x^2+0x+3) \\ \hline 2x+1 \end{array}$$

So remainder is $2x+1$

$\Rightarrow$ in $\mathbb{Z}_5[x]/(x^2+1)$

$[(3x+4)(x+1)] = [2x+1]$