Vanderpool

TMath 402

Autumn 2022

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] (§2.2) Let a and b be integers and p be a prime number. If p divides ab, then either p divides a or p divides b.

2. [3] (§3.2) All groups are abelian/commutative.

3. [3] (§5.1) Let $\sigma \in S_{12}$. Then $\sigma^{12} = ()$.

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

4. [9] For each of the sets and operations below, determine if they define a group. If no, briefly explain why. If yes, briefly describe the process you used to reach that answer.

Sets S & Operator $\star$	Is a Group?
a) $\mathbb{Z}_{10}$ $\star$ is multiplication under modulo 10	
b) $\{a, b, c, d\}$ $\star$ is defined by	
c) Set generated by r and s subject to the relations $e = r^{10} = s^2$ and $srs = r^{-1}$ where e is the identity.	

$\star$	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	c	d	b	a
d	d	c	a	b

5. [2] Choose a set (any set you want!) and define a (non-trivial) equivalence relation on it. (Non-trivial means that every element is not in its own equivalence class nor that all elements are in the same equivalence class.)

6. Consider the integers $\mathbb{Z}$ with the the binary operator of addition. Let α be an integer. Define $\alpha\mathbb{Z} = \{\alpha k | k \in \mathbb{Z}\}$.

   (a) [4] (§3.3) Show $5\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.

   (b) [3] (ModuloActivity) Draw (a partial) Cayley Diagram for $5\mathbb{Z}$.

   (c) [2] (4.1) Find a subgroup of $5\mathbb{Z}$.

   (d) [3] Create a partial subgroup lattice for $\mathbb{Z}$. Include $5\mathbb{Z}$ and your answer to part (c) in the lattice.

7. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
   No, doing both questions will not earn you extra credit.

   Theorem 1. *Let G be a group. Prove that $ba = ca$ implies $b = a$ and $ab = ac$ implies $b = c$ for all elements a, b, and c in G.*

   Theorem 2. *Let G be a finite cyclic group with order* 20 *generated by g . Prove g^n generates G if and only if* $\gcd(n, 20) = 1$.