Take Home Final

TMath 402

Autumn 2022

This section is to be taken home, completed, and turned in by 12:10am on Dec 14. There is no time limit and you do not need to type up your solutions to get full marks although the answers should use correct symbols and be well edited.

You may discuss this problem with anyone else from the class and use the class resources posted on Canvas. You may not consult anyone or any resource that is not affiliated with the class such as tutors, websites, or other textbooks.

1. [4] Clearly define a set S along with appropriate binary operator(s) that satisfy the following. Briefly justify how each condition is met.
   - The set S has more than 5 elements,
   - forms a non-Abelian group with a binary operator (we will denote with $+$),
   - has at least one subgroup that is not normal.
2. Let G be the group in C^* generated by $\omega = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$. Define the homomorphism $h : G \to G$ by $h(\omega) = \omega^3$.
   (a) [2] Build the Cayley Diagram or Cayley Table for G.
   (b) [3] Find the map ϕ promised in the 1st isomorphism theorem from $G/\ker(h) \to G$.
   (c) [2] Build the subgroup lattice for G and the subgroup lattice for $G/\ker(h)$.
3. Consider Dr. Vanderpool's proof written for the theorem below.
   - [2] Identify any logical errors. (at least 2)
   - [2] Provide concrete suggestions for how the proof can be improved. (at least 2)

Theorem 1. *Let G and H be groups and let $\phi : G \to H$ be a homomorphism. Define a relation, $\sim$, on G by $a \sim b$ if $\phi(a) = \phi(b)$ for all a and b in G. Prove $\sim$ is an equivalence relation and the equivalence classes are the cosets of* $\ker(\phi)$.

Proof. Clearly $\phi(a) = \phi(a)$ and if $\phi(a) = \phi(b)$ then $\phi(b) = \phi(a)$ for all $b \in G$ so $\sim$ is symmetric and reflexive. Also, if $\phi(a) = \phi(b)$ and $\phi(b) = \phi(c)$ then transitivity of the equal sign in H implies $\phi(a) = \phi(c)$. Therefore $\sim$ is an equivalence relation.

To see the equivalence classes of $\sim$, let $G = \mathbb{Z}$ and $H = \mathbb{Z}_8$. Define $\phi : \mathbb{Z} \to \mathbb{Z}_8$ by $\phi(1) = 1$. Notice that ϕ is a homomorphism since

$$\begin{aligned}\phi(a)\phi(b) &= \phi(1 + ... + 1)\phi(1 + 1.. + 1) \text{ since } a = 1 + ... + 1, a \text{ times}\\ &= (1 + ... + 1) + (1 + 1 + ... + 1) \text{ be definition of } \phi\\ &= (1 + 1 + ... + 1) + (1 + ... + 1) \text{ re-associate}\\ &= \phi(b)\phi(a).\end{aligned}$$

We assume that $a \sim b$ in G and will show that a is in the coset $b\ker(\phi)$. Since $a \sim b$ we know that $\phi(a) = \phi(b)$. Apply $[\phi(b)]^{-1}$ to both sides then $\phi(a)[\phi(b)]^{-1} = e_G$ where e_G is the identity in G. Since ϕ is a homomorphism $[\phi(b)]^{-1} = \phi(b^{-1})$ and $\phi(a)\phi(b^{-1}) = \phi(ab^{-1})$. This we have $\phi(ab^{-1}) = e_G$. This means that $\phi(ab^{-1}) \in \ker(\phi)$. Let $k = ab^{-1}$, so we know $k \in \ker(\phi)$. Therefore $ab^{-1} = k$ or $a = kb$ meaning that $a \in b\ker(\phi)$ or that a is in the coset $b\ker(\phi)$ which is what we wanted to show. $\square$