Key

Vanderpool

TMath 402

Autumn 2022

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] Let G be a group and let a, b, and c be in G. If $ba = ca$, then $b = c$.

(+1) True Since G is a group $a^{-1} \in G$ so

logic (+1)
def/notations (+1)

$ba = ca$

$\Rightarrow (ba)a^{-1} = (ca)a^{-1}$ by applying a^{-1} on the right

$\Rightarrow b(aa^{-1}) = c(aa^{-1})$ since groups are associative

$\Rightarrow b \cdot e = c \cdot e$ since $aa^{-1} = e$ which is the identity in G

$\Rightarrow b = c$ ✓

2. [3] Let $\phi : G_1 \to G_2$ be a homomorphism between groups. If e_1 is the identity in G_1, then $\phi(e_1)$ is the identify in G_2.

(+1) True Let e_2 be the identity in G_2, we want to show $e_2 = \phi(e_1)$.

Consider $e_2\phi(e_1) = \phi(e_1)$ b/c e_2 is the identity in G_2

logic (+1)
def/notations (+1)

$= \phi(e_1 \cdot e_1)$ since e_1 is the identity in G_1

$= \phi(e_1)\phi(e_1)$ b/c ϕ is a homomorphism.

Since $e_2\phi(e_1) = \phi(e_1)\phi(e_1)$ and G_2 is a group we can apply $[\phi(e_1)]^{-1}$ on the right to get $e_2 = \phi(e_1)$ ✓

3. [3] If H is a normal subgroup of a group G, then $gh = hg$ for all $h \in H$ and $g \in G$.

(+1) False The normal subgroup definition is for the set H, that is $gH = Hg$ $\forall g \in G$.

def/notations (+1)
counterex (+1)

Counterexample: Consider $D_4 = \langle r, f \rangle$ and $H = \langle r \rangle = \{r, r^2, r^3, e\}$

note $|H| = 4 \Rightarrow [D_4 : H] = 2 \Rightarrow H \trianglelefteq D_4$.

Let $f \in D_4$ and $r \in H$ & notice $fr \neq rf$

in fact $fr = r^3 f$.

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

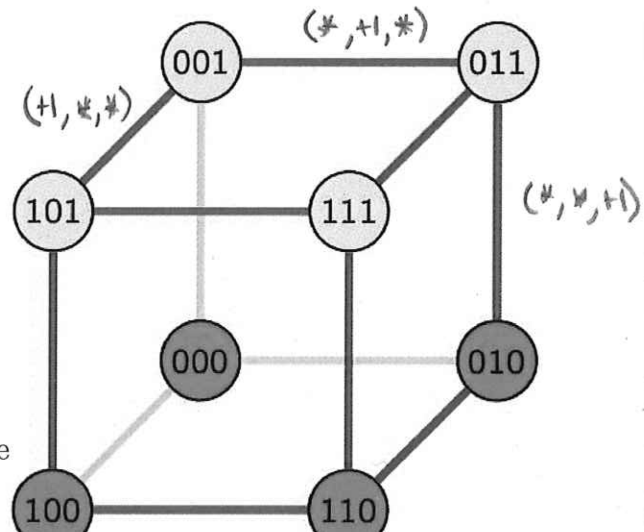
4. [3] Find two non-isomorphic groups of order 6. Explain why the groups are not isomorphic.

$\mathbb{Z}_6$ is an abelian group

S_3 is a non abelian group

groups of order 6 (+1)
reasoning/logic (+1)
notation/def (+1)

5. The Cayley diagram for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is below.



(a) [1] What is the identify element?

$(0,0,0)$

(b) [2] What action/group operation corresponds to the blue arrow/line?

adding 1 in the first coordinate (mod 2)

(c) [3] Four elements are identified as green in the figure and form a subgroup H. You can assume H is normal in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
Find the cosets of H.

$H = \{(0,0,0), (1,0,0), (1,1,0), (0,1,0)\}$

$(0,0,1) + H = \{(1,0,1), (1,0,1), (1,1,1), (0,1,1)\}$

(d) [3] Show $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/H$ is isomorphic to $\mathbb{Z}_2$.

Define

$$\begin{array}{ccc} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \xrightarrow{h} & \mathbb{Z}_2 \\ (a,b,c) & \longmapsto & c \\ \downarrow & & \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2/H & & \end{array}$$

note $\ker h = H$ and h is onto
$\therefore$ by the 1^{st} iso thm $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2/H \cong \mathbb{Z}_2$

OR we will write the Cayley Table for $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/H$ so we can write the isomorphism explicitly

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2/H$	H	$(0,0,1)+H$
H	H	$(0,0,1)+H$
$(0,0,1)+H$	$(0,0,1)+H$	H

(+1) def of map
(+.5) onto
(+.5) one-to-one
(+.5) homomorp
(+.5) correct elem

So this iso

	$\mathbb{Z}_2$
$H \longmapsto$	0
$(0,0,1)+H \longmapsto$	1

should have had 5 elements ✗ kicking self ✗

6. Let $X = \{a, b, c, d\}$ and let S be the set of all permutations on X.

(a) [4] Let $\alpha = (a\,b\,c\,d)$ and $\beta = (a\,c)$. Find the following:

i. α^{-1}

$(a\,d\,c\,b)$ (+1)

Check: $(a\,b\,c\,d)(a\,d\,c\,b) = (a)(b)(c)(c)$ ✓

ii. $\beta\alpha$

$(a\,c)(a\,b\,c\,d) = (a\,b)(c\,d)$ (+1)

def of comp (+.5)

iii. Does α and β commute? Explain.

$\alpha\beta = (a\,b\,c\,d)(a\,c) = (a\,d)(b\,c)$ Nope. ? (+.5)

Switch order of elem (+.5) ✗ answer from (a ii) (+.5)

(b) [2] Find an element of S that has order 6 (if such an element exists).

order of elements (+.5) cases (all) (+.5) logic (+1)

Note in cycle notation $\sigma \in S$ could look like $(\alpha\,\beta\,\gamma\,\delta)$ with order 4, $(\alpha\,\beta\,\gamma)$ with order 3, $(\alpha\,\beta)(\delta\,\gamma)$ with order 2, $(\alpha\,\beta)$ with order 2, or the identity.

note ~~(a)(b)(c)(d)~~ $[(ab)(dce)]^2 =$

Thus by exhaustion ∄ element of order 6.

(c) [4] Define a subset $H = \{\sigma \in S | \sigma(b) = b\}$. That is, all permutations that does not move the letter b. For example, β from part (a) is in H. Show that H is a subgroup of S.

We verify the conditions of a subgroup:

Closure: Let $\delta, \gamma \in H$ then $\delta(1) = 1 = \gamma(1)$.
Note $(\delta\cdot\gamma)(1) = \delta(\gamma(1)) = \delta(1) = 1$
thus $\delta\cdot\gamma \in H$.

Identity: The identity permutation does not move any elements, so certainly not 1.
∴ identity $e \in H$

Inverses: If $\delta \in H$ we know $\delta(1) = 1$.
Notice $\delta^{-1} \in G$ b/c G is a group and since $\delta(1) = 1$, $\delta^{-1}(1) = 1$ ∴ $\delta^{-1} \in H$.

We inherit associativity from G thus H is a subgroup of G. //

We verify 1) $H \neq \emptyset$ and 2) if $f, g \in H$, then $fg^{-1} \in H$.

1) Notice the identity permutation does not move any elements, so 1 is not moved
$\Rightarrow$ the identity $e \in H$. ∴ $H \neq \emptyset$.

2) Let $f, g \in H$. Since G is a group $g^{-1} \in G$.
Note that $g(1) = 1$ b/c $g \in H$, thus $g^{-1}(1) = 1$.
Consider $(f\cdot g^{-1})(1) = f(g^{-1}(1))$
$= f(1)$ by above
$= 1$ b/c $f \in H$

Thus $f\cdot g^{-1} \in H$. //

def of subgp/ prop (+1)

for each condition +.5 +.5 +.5 +.5

understand def/ex (+.5) notation (+.5)

4	3	2	1
well written	good but some math errors or writing that needs addressing	good intuition but at least 1 serious flaw	I don't understand but I see you worked.

7. [4] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit. No, doing both questions will not earn you extra credit.

normal Writing Rubric

note: probably did not need 1st iso thm for but I liked it :)

Theorem 1. *Let m and n be integers. If $\gcd(m, n) = 1$, then $\mathbb{Z}_n \times \mathbb{Z}_m$ is isomorphic to $\mathbb{Z}_{nm}$.*

Theorem 2. *Every finite cyclic group is isomorphic to $\mathbb{Z}_n$ for some integer n.* (from Written HW6)

Thm 1:

Pf: I'll use the 1st iso thm b/c I love it.

Define $h: \mathbb{Z} \to \mathbb{Z}_n \times \mathbb{Z}_m$.

$1 \mapsto (1,1)$

Notice h is defined on the generator of $\mathbb{Z}$ so can be extended to form a homomorphism.

Consider $\ker h = \{a \in \mathbb{Z} \mid h(a) = (0,0)\}$

$= \{a \in \mathbb{Z} \mid a \equiv 0 \bmod n$ AND $a \equiv 0 \bmod m\}$

Since $\gcd(m,n) = 1$

$= (mn)\mathbb{Z}$.

To show h is onto, we verify the generators $(1,0)$ and $(0,1)$ are in the image of h.

Since $\gcd(m,n) = 1$ $\exists\, l, k \in \mathbb{Z} \ni$

$mk + nl = 1$ or $mk = 1 - nl$.

Notice $h(mk) = h(1 - nl) = (1,0)$

and $h(nl) = h(1 - mk) = (0,1)$.

The first 1st iso. thm then can be used with

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "h"] \arrow[d, "\phi"'] & \mathbb{Z}_n \times \mathbb{Z}_m \\ \mathbb{Z}/(mn)\mathbb{Z} \arrow[u, phantom, "\,"] \arrow[ur, dashed, "\cong"'] & \end{tikzcd}$$

$\phi: \mathbb{Z} \to \mathbb{Z}/(mn)\mathbb{Z}$

$a \mapsto a \bmod (mn)$

We have an isomorphism $\cong$ from $\mathbb{Z}/(mn)\mathbb{Z}$ to $\mathbb{Z}_n \times \mathbb{Z}_m$

or from $\mathbb{Z}_{mn}$ to $\mathbb{Z}_n \times \mathbb{Z}_m$ //

Thm 2:

Pf: Let G be a finite cyclic group so $G = \langle g \rangle$. We will use the 1st iso thm.

Consider $h: \mathbb{Z} \to G$.

$a \mapsto g^a$

Note h is a homomorphism since

$h(a+b) = g^{a+b} = g^a g^b = h(a)h(b)$

$\forall a, b \in \mathbb{Z}$

Consider $\ker h = \{a \in \mathbb{Z} \mid h(a) = e\}$ where e is the identity in G.

$\ker h = \{a \in \mathbb{Z} \mid g^a = e\}$

$= \{a \in \mathbb{Z} \mid a \equiv 0 \bmod n\}$

(where n is the order of g)

$= n\mathbb{Z}$

Notice h is onto G.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "h"] \arrow[d, "\phi"'] & G \\ \mathbb{Z}/n\mathbb{Z} \arrow[ur, dashed] & \end{tikzcd}$$

Let $\phi: \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$

$a \mapsto a \bmod n$

The 1st iso. Thrm $\Rightarrow$ $\exists$ an isomorphism $\cong$ from $\mathbb{Z}/n\mathbb{Z}$ to G

or from $\mathbb{Z}_n$ to G

since $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$. //