Vanderpool

TMath 402

Autumn 2022

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] Let G be a group and let a, b, and c be in G. If $ba = ca$, then $b = c$.

2. [3] Let $\phi : G_1 \to G_2$ be a homomorphism between groups. If e_1 is the identity in G_1, then $\phi(e_1)$ is the identify in G_2.

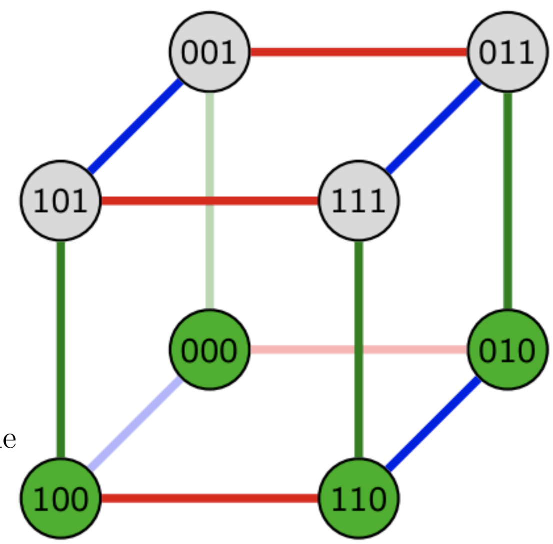
3. [3] If H is a normal subgroup of a group G, then $gh = hg$ for all $h \in H$ and $g \in G$.

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

4. [3] Find two non-isomorphic groups of order 6. Explain why the groups are not isomorphic.

5. The Cayley diagram for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is below.

   (a) [1] What is the identify element?

   (b) [2] What action/group operation corresponds to the blue arrow/line?

   (c) [3] Four elements are identified as green in the figure and form a subgroup H. You can assume H is normal in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
   Find the cosets of H.

   (d) [3] Show $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/H$ is isomorphic to $\mathbb{Z}_2$.

6. Let $X = \{a, b, c, d\}$ and let S be the set of all permutations on X.

   (a) [4] Let $\alpha = (a\,b\,c\,d)$ and $\beta = (a\,c)$. Find the following:

      i. α^{-1}

      ii. $\beta\alpha$

      iii. Does α and β commute? Explain.

   (b) [2] Find an element of S that has order 6 (if such an element exists).

   (c) [4] Define a subset $H = \{\sigma \in S | \sigma(b) = b\}$. That is, all permutations that does not move the letter b. For example, β from part (a) is in H. Show that H is a subgroup of S.

7. [4] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

Theorem 1. *Let m and n be integers. If $\gcd(m, n) = 1$, then $\mathbb{Z}_n \times \mathbb{Z}_m$ is isomorphic to $\mathbb{Z}_{nm}$.*

Theorem 2. *Every finite cyclic group is isomorphic to $\mathbb{Z}_n$ for some integer n.*