MIDTERM Key TMATH 308 SPRING 2020

Once you have completed the exam, consider reflecting on your performance. The point value for each question is indicated inside a bracket at the start of each problem. The total exam is worth 45 points. You are to guess what your final score on this exam will be (either figuring it out question by question or with a holistic determination). The student(s) closest to their raw earned score will receive 1 bonus point. There is no penalty for guessing.

1. True/False: If the statement is *always* true, give a *brief* explanation of what it is (not a "formal proof"). If the statement is false, give a counterexample.

DHW §3.2 #2

(a) [4] The set $X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + x_2 + x_3 \text{ is odd} \right\}$ is a subspace of $\mathbb{R}^3$.

false (+1)

nonempty b/c $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in X$ ✓

closed under addition? No!

start (+.5)

logic (+1)

sense/style (+.5)

def of subspace (+1)

note $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

but $0+0+2$ is not odd so $(0\ 0\ 2)^T \notin X$

DHW §1.4 #3

(b) [4] Given $n \times n$ matrices A and B, if AB equals the zero matrix, then either A or B is the zero matrix.

false (+1)

Consider $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

start (+.5)

logic (+1)

sense/style (+.5)

Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

matrix mult/zero matrix (+1)

even though neither A or B

WHW §3.2 #17

(c) [4] Let $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ be a spanning set for a vector space $\mathbb{R}^3$ and $\vec{v_4} \in \mathbb{R}^3$. Then $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ is a spanning set for the vector space $\mathbb{R}^3$ as well.

True (+1)

To show $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a spanning set for $\mathbb{R}^3$

start (+.5)

we need to show for any $\vec{z} \in \mathbb{R}^3$ that $\exists$ c_1, c_2, c_3, c_4

logic (+1)

$\ni \vec{z} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$.

sense/style (+.5)

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbb{R}^3$ $\exists$ b_1, b_2, b_3 $\ni$

span set def (+1)

$\vec{z} = b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3 = b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3 + 0 \cdot \vec{v}_4$

we thus have the scalars. :)

2. [8] Assume that x, y, and z are nonzero real numbers.

DHW §1.3 #12

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & x \\ y & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & y \\ x & 2 \\ 5 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 & 0 & 1 \\ z & 0 & 0 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} -1 \\ 0 \\ z \end{bmatrix}$.

Perform each of the following computations or indicate that the operations is not defined:

notation (+1.5)

(a) B^T

$$\begin{bmatrix} -1 & x & 5 \\ y & 2 & 1 \end{bmatrix}$$

switch rows & columns (+.5)

got it (+.5)

(b) $A+B$

3x2 size + 3x2 size should work? (+.5)

$$\begin{bmatrix} 1 & 3 \\ 0 & x \\ y & 2 \end{bmatrix} + \begin{bmatrix} -1 & y \\ x & 2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3+y \\ x & 2+x \\ 5+y & 3 \end{bmatrix}$$

got it (+.5)

(c) AC

3x2 times 2x4 should give a 3 by 4 (+.5)

mult alg (+1)

$$\begin{bmatrix} 1 & 3 \\ 0 & x \\ y & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ z & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2+3z & -3 & 0 & 7 \\ xz & 0 & 0 & 2x \\ 2y+2z & -3y & 0 & y+4 \end{bmatrix}$$

got it (+.5)

notation (+.5)

(d) CA

2x4 times 3x2 will not be defined? (+1)

(recall the # of col in the left matrix must equal # of rows in the right matrix to multiply)

(e) $\det(C)$

(+1) C is a 2 by 4 matrix but determinants are only defined for square matrices

3. Consider the augmented matrix $X = \left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 & -1 & 0 & b-c \\ 0 & 0 & 0 & 0 & 0 & 0 & c+a-b \end{array}\right]$

(a) [3] What minimal restriction are required on scalars a, b, and c to make X *consistent.*

need $c+a-b=0$ (+1)

or $a=b-c$

the above system would become

$\left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 & -1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

start (+.5)
sense (+1)
on topic (+.5)

(b) [5] Assume that $a = b = c = 0$, write the solution set for the associated system of equations defined by X.

use 0 (+.5)
set up free variables (+1)
interpret as system (+1)
write x_1 & x_4
notation (+1)
sense (+.5)

So considering

$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ \hline 1 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$

x_2, x_3, x_5 & x_6 are free variables. Call these s, t, u and v respectively.

Then $x_1 - s + 2t + v = 0$ & $x_4 - u = 0$

$\Rightarrow x_1 = s - 2t - v$ and $x_4 = u$

$\Rightarrow \left\{ \begin{bmatrix} s-2t-v \\ s \\ t \\ u \\ u \\ v \end{bmatrix} \middle| \, s,t,u,v \in \mathbb{R} \right\}$ → got it. (+.5)

answer is a set (+.5)

4. Let $A = \begin{bmatrix} 3 & 6 & 12 \\ 1 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) [3] Determine the elementary row operations to reduce the matrix A to B.

$\begin{bmatrix} 3 & 6 & 12 \\ 1 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

DHW4 §1.5 #3

(b) [3] Then find the matrix E, a product of elementary matrices, so that $EA = B$. *You may leave E in factored form.*

(+.5) correspond to (a)

$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(+.5)} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(+.5)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{(+.5)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(+.5)}$

order (+.5)

DHW5 §2.1 #3
DHW6 §2.2 #9

(c) [2] Find the determinant of E.

(+1) since $\det(AB) = \det(A)\det(B)$ we can find the determinant of the factor matrices above & then just multiply !!

$\underbrace{(1)(-1)(1)(\tfrac{1}{2})(\tfrac{1}{3})}_{(+.5)} = -\tfrac{1}{6}$ got it (+.5) justification for each is stated (+.5)

Midterm Review on Day

start (+.5)
sense (+.5)

5. [5] Grade the following proof on clarity (scale between 0 and 3) and accuracy/correctness (scale between 0 and 4). Justify your scores by indicating what was done well or what can be improved upon (it's likely that there will be a mix of such comments!).

good/accurate connection made here

Let A and B be $n \times n$ invertible matrices, we will show that AB is nonsingular.

good intro/outline

Recall that a matrix AB is nonsingular if the inverse, $(AB)^{-1}$ exists. To complete the proof thus suffices to show that $(AB)^{-1}$ exists.

→ need to say A^{-1} exists & what is

Since we are assuming that A and B are invertible matrices, we know $AA^{-1} = A^{-1}A = I$ and that $BB^{-1} = B^{-1}B = I$, $\boxed{\text{thus } A \text{ and } B \text{ commute.}}$ → Does not follow?

Consider the element $A^{-1}B^{-1}$. We will show $A^{-1}B^{-1}$ is the inverse to AB. We compute:

could make this much easier to read.

$(AB)(A^{-1}B^{-1}) = ABA^{-1}B^{-1}$(since matrix multiplication is associative)
$\cancel{ABA^{-1}B^{-1}} = BAA^{-1}B^{-1}$(since we know A and B commute)
$\cancel{B(AA^{-1})B^{-1}} = BIB^{-1}$(since $AA^{-1} = I$)
$\cancel{BIB^{-1}} = BB^{-1}$(since I is the multiplicative identity)
$\cancel{BB^{-1}} = I$(by definition of B^{-1})

like the words linked to the algebraic steps

good conclusion

Thus we have shown that $A^{-1}B^{-1}$ is the inverse to AB meaning that AB is nonsingular, which is what we wanted to show. (+.5)

(+.5)

Accuracy/correctness 3
Definition of inverse is correct
Matrix multiplication is not generally commutative - you showed that A and A^{-1} commute and that B + B^{-1} commute. Nothing was shown about A+B!

logic flow (+.5)

specific ex. (+.5)

Clarity 2 or 3
lots of narration? :)
remember to introduce terms before you use them (like A^{-1})
consider lining up "=" signs & only writing the right side

6. [4] Take one (mathematical, technical, or social!) problem you have worked on this quarter that you struggled to understand and solve, and explain how the struggle itself was valuable.

Midterm Review Day!!

start (+1)
describe problem (+1)
struggle usefulness (+1)
sense (+1)