Once you have completed the exam, consider reflecting on your performance. The point value for each question is indicated inside a bracket at the start of each problem. The total exam is worth 45 points. You are to guess what your final score on this exam will be (either figuring it out question by question or with a holistic determination). The student(s) closest to their raw earned score will receive 1 bonus point. There is no penalty for guessing.

1. True/False: If the statement is *always* true, give a *brief* explanation of what it is (not a "formal proof"). If the statement is false, give a counterexample.

   (a) [4] The set $X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + x_2 + x_3 \text{ is odd} \right\}$ is a subspace of $\mathbb{R}^3$.

   (b) [4] Given $n \times n$ matrices A and B, if AB equals the zero matrix, then either A or B is the zero matrix.

   (c) [4] Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ be a spanning set for a vector space $\mathbb{R}^3$ and $\overrightarrow{v_4} \in \mathbb{R}^3$. Then $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}\}$ is a spanning set for the vector space $\mathbb{R}^3$ as well.

2. [8] Assume that x, y, and z are nonzero real numbers.

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & x \\ y & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & y \\ x & 2 \\ 5 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 & 0 & 1 \\ z & 0 & 0 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} -1 \\ 0 \\ z \end{bmatrix}$.

Perform each of the following computations or indicate that the operations is not defined:

(a) B^T

(b) $A + B$

(c) AC

(d) CA

(e) $\det(C)$

3. Consider the augmented matrix $X = \left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 & -1 & 0 & b-c \\ 0 & 0 & 0 & 0 & 0 & 0 & c+a-b \end{array}\right]$

(a) [3] What minimal restriction are required on scalars a, b, and c to make X *consistent.*

(b) [5] Assume that $a = b = c = 0$, write the solution set for the associated system of equations defined by X.

4. Let $A = \begin{bmatrix} 3 & 6 & 12 \\ 1 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) [3] Determine the elementary row operations to reduce the matrix A to B.

(b) [3] Then find the matrix E, a product of elementary matrices, so that $EA = B$. *You may leave E in factored form.*

(c) [2] Find the determinant of E.

5. [5] Grade the following proof on clarity (scale between 0 and 3) and accuracy/correctness (scale between 0 and 4). Justify your scores by indicating what was done well or what can be improved upon (it's likely that there will be a mix of such comments!).

   Let A and B be $n \times n$ invertible matrices, we will show that AB is nonsingular.

   Recall that a matrix AB is nonsingular if the inverse, $(AB)^{-1}$ exists. To complete the proof thus suffices to show that $(AB)^{-1}$ exists.

   Since we are assuming that A and B are invertible matrices, we know $AA^{-1} = A^{-1}A = I$ and that $BB^{-1} = B^{-1}B = I$, thus A and B commute.

   Consider the element $A^{-1}B^{-1}$. We will show $A^{-1}B^{-1}$ is the inverse to AB. We compute:

   $(AB)(A^{-1}B^{-1}) = ABA^{-1}B^{-1}$(since matrix multiplication is associative)
   $ABA^{-1}B^{-1} = BAA^{-1}B^{-1}$(since we know A and B commute)
   $B(AA^{-1})B^{-1} = BIB^{-1}$(since $AA^{-1} = I$)
   $BIB^{-1} = BB^{-1}$(since I is the multiplicative identity)
   $BB^{-1} = I$(by definition of B^{-1})

   Thus we have shown that $A^{-1}B^{-1}$ is the inverse to AB meaning that AB is nonsingular, which is what we wanted to show.

6. [4] Take one (mathematical, technical, or social!) problem you have worked on this quarter that you struggled to understand and solve, and explain how the struggle itself was valuable.