Vanderpool

# TMath 403

Spring 2024

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] If p is prime, then there is only one finite ring of order p.

2. [3] Let $R$ be a ring and $x \in R$. If $x$ is not a unit, then $x$ is a zero divisor.

3. [3] Let $R$ be a ring with unity/one. Show if $\phi : R \to S$ is a ring homomorphism then $\phi(1)$ is idempotent.

4. [8] For each of the terms below, determine if the term is used to describe an element, a set, both, or neither. Then provide examples for each.

| | element? | set? |
|---|---|---|
| abelian | no | yes,<br>groups work<br>$C_5 = \{r \mid r^5 = 1\}$ |
| unit | | |
| zero divisor | | |
| kernel | | |
| prime | | |

5. Consider $R = \mathbb{Z}_9 \times \mathbb{Z}_3$.

   (a) [3] Find an ideal $I$, so that $R/I$ is a ring but not a field. Justify your answer.

   (b) [3] Find an ideal $I$, so that $R/I$ is a field. Justify your answer.

6. Use the first three letters of your first name to build a polynomial of the form $a_0 + a_1x + a_2x^2$ in $\mathbb{Z}_3[x]$. Specifically, use the table below to let $a_0$ be the number that corresponds to the first letter in your first name. For Ruth then "R" would set $a_0 = 0$. Let $a_1$ be the number that corresponds to the second letter and $a_2$ correspond to the third letter. For Ruth then $a_1 = 0$ and $a_2 = 2$, thus the polynomial for Ruth is $0 + 0x + 2x^2$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | A | D | G | J | M | P | S | V | Y |
| 2 | B | E | H | K | N | Q | T | W | Z |
| 0 | C | F | I | L | O | R | U | X | |

(a) [1] Let $p(x)$ represent the polynomial of the form $a_0 + a_1x + a_2x^2$ corresponding with your first name. Write down $p(x)$.

(b) [2] Find a representative of $x^3$ in $\mathbb{Z}_3[x]/(p(x))$ with degree less than 2.

(c) [4] how many elements does $\mathbb{Z}_3[x]/(p(x))$ have? Justify your answer.

7. Consider:

**Theorem 1.** *Let $I$ and $J$ be ideals in a ring $R$. Then $I \cap J$ is an ideal in $R$.*

(a) [2] Find an example $I$, $J$, and $R$ that helps verify Theorem 1.

(b) [8] Prove Theorem 1.