Key

Exam 1 | TMath 126 | Autumn 2024

Va ~~As a reminder, you are welcome to use a two-sided 3.5" by 5" index card with notes (written or typed)~~, a non-internet accessing calculator (which includes Desmos Test Mode) but no books, other notes, or peers.

1. [9] TRUE/FALSE: Write True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, write False and provide a counterexample or brief justification.

   (a) (Quiz2#1) If $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^3$ so that $\vec{v} \cdot \vec{w} \neq 0$ (that is, the dot product of vectors v and w), then $\vec{v}$ and $\vec{w}$ are not perpendicular or parallel.

   (+.5) False $\quad \vec{v}$ and $\vec{w}$ could be parallel.

   ex $\vec{v} = \langle 1,0,0 \rangle$ and $\vec{w} = \langle 2,0,0 \rangle$

   notice $\vec{v}$ is parallel to $\vec{w}$ and

   $\vec{v} \cdot \vec{w} = \langle 1,0,0 \rangle \cdot \langle 2,0,0 \rangle = 2+0+0 \neq 0$

   definition of dot (+1)
   start (+.5)
   sense (+.5)
   counterex (+.5)

   (b) (§13.2#26) If $\vec{r}(t) = \langle 3^t, t\cos(2t), t^3 - 3t \rangle$, then the line tangent to $\vec{r}(1)$ is:

   $$\langle x, y, z \rangle = \langle 1, 1, -2 \rangle + \langle 3^t(\ln 3), -2t\sin(2t) + \cos(2t), 3t^2 - 3 \rangle$$

   line def (+1)
   ~~sense~~ (+1)
   start (+.5)

   $\vec{r}(1) = \langle 3^1, 1\cos(2), 1^3 - 3\cdot 1 \rangle$

   $= \langle 3, -.416, -2 \rangle$ eval +.5 derivatives +1.5 sense +1

   hm...

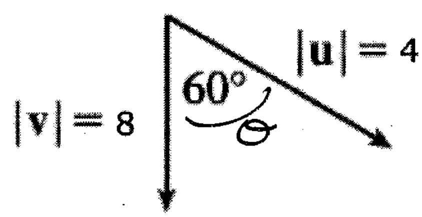
   This is not a line? False ? (+.5)

   Lines look like $\langle x_0, y_0, z_0 \rangle + t\vec{v} = \langle x, y, z \rangle$ where $\vec{v}$ is a directional vector with numbers as entries.

   This $\vec{v}$ is a function of t...

   Not True false

   ~~(c)~~ (WebHW12.4 #3) Find $\vec{u} \times \vec{v}$ given the information on the right.



   notation (+.5)

   Recall $\vec{u} \times \vec{v}$ is a vector

   (+.5) $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \sin\theta$

   (+1) $= 4 \cdot 8 \cdot \sin(60°) = 32\left(\frac{\sqrt{3}}{2}\right) = 16\sqrt{3} \approx 27.7$

   The question wants a vector so we need a direction too

   (+1) Right hand rule => into the paper

   So $\vec{u} \times \vec{v}$ has magnitude 27.7 into the paper

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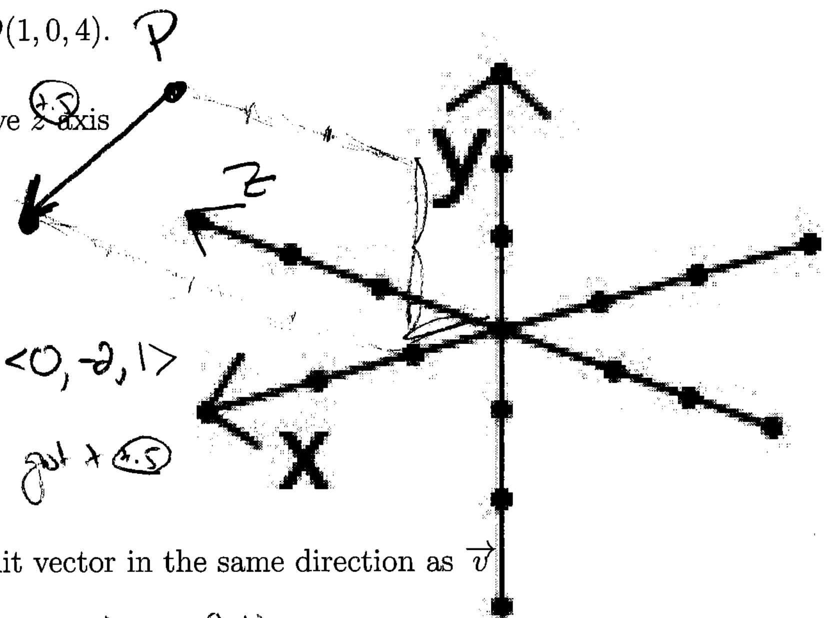
Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Consider the points $P(1,2,3)$ and $Q(1,0,4)$.
   Let $\vec{v} = \langle 0,-2,1\rangle$.

   (a) [2] (Quiz1#1) Label the positive (+.5) ~~z~~ axis and then plot the vector $\overrightarrow{PQ}$

   P(+.5) Q(+.5) direction(+.5)



   (b) [1] (PracticeExam1#2) Find the components of $\overrightarrow{PQ}$.

   $< 1-1, 0-2, 4-3 > = <0,-2,1>$

   subtraction (+.5) gut + (+.5)

   (c) [2] (WebHW12.2#7) Find a unit vector in the same direction as $\vec{v}$

   $$\frac{\vec{v}}{\|\vec{v}\|} = \frac{<0,-2,1>}{\sqrt{0^2+(-2)^2+1^2}} \text{ (+1)} = \frac{<0,-2,1>}{\sqrt{5}} = <0, \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}>$$

   (+.5)

   notation (+.5)

   (d) [3] (WebHW12.3#6) Find the angle $\overrightarrow{PQ}$ makes with $\vec{v}$.

   (+.5) [Recall $\overrightarrow{PQ} \cdot \vec{v} = \|\overrightarrow{PQ}\| \, \|\vec{v}\| \cos\theta$ where θ is angle between

   (+1) (+.5)

   $<0,-2,1> \cdot < 0,-2,1> = \sqrt{0^2+(-2)^2+1^2}\,\sqrt{0^2+(-2)^2+1^2} \cos\theta$

   $0^2+(-2)^2+1^2 = \sqrt{5}\sqrt{5} \cos\theta$

   alg to solve (+1)

   $\Rightarrow \frac{5}{5} = \cos\theta \Rightarrow 1 = \cos\theta \Rightarrow \theta = 0°$ or ~~180°~~

   so parallel

   would be in opposite directions

   (e) [3] (Quiz2 #1) Find an equation of a plane passing through $(0,-2,1)$ and normal/orthogonal/perpendicular to $\vec{v}$ LOTS of answers for this?

   start (+.5)

   (+.5) equation of plane

   notation (+1)

   $0 = \vec{v} \cdot (<x,y,z> - <0,-2,1>)$

   (+.5) (+.5)

   $0 = <0,-2,1> \cdot (<x,y,z> - <0,-2,1>)$

   $0 = <0,-2,1> \cdot <x, y+2, z+1>$

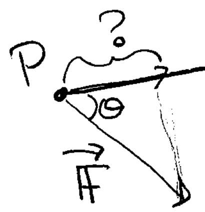
   $0 = -2(y+2) + 1(z+1)$

   $0 = -2y - 4 + z + 1$

   $3 = z - 2y$

\\

3. [3] (WrittenHW12.3 #50) A force is given by a vector $\overrightarrow{F} = 3\overrightarrow{i} + 4\overrightarrow{j} + 5\overrightarrow{k}$ and moves a particle from $P(2,1,0)$ to $Q(4,6,2)$. Find the work done.



(+1) $\langle 4-2, 6-1, 2-0 \rangle$

$\langle 2,5,2 \rangle$

Sohcahtoa $\cos\Theta = \dfrac{?}{\|\vec{F}\|}$

$\Rightarrow ? = \|\vec{F}\| \cos\Theta$

need the component of $\vec{F}$ in direction of $\vec{PQ}$.

Work = (Force in direction of PQ) · (length of PQ)

= ? · ‖PQ‖

Work = $\|\vec{F}\|\cos\Theta \cdot \|PQ\|$

Work = $\|\vec{F}\| \, \|\vec{PQ}\| \cos\Theta$

(+1) Work = $\vec{F} \cdot \vec{PQ}$

would have been nice if I remembered that

$\Rightarrow \langle 3,4,5 \rangle \cdot \langle 2,5,2 \rangle = 6+20+10$

$= 36$ (+1)

4. (WordProblem #2) A plane is flying at 90 knots 16° north from due east.

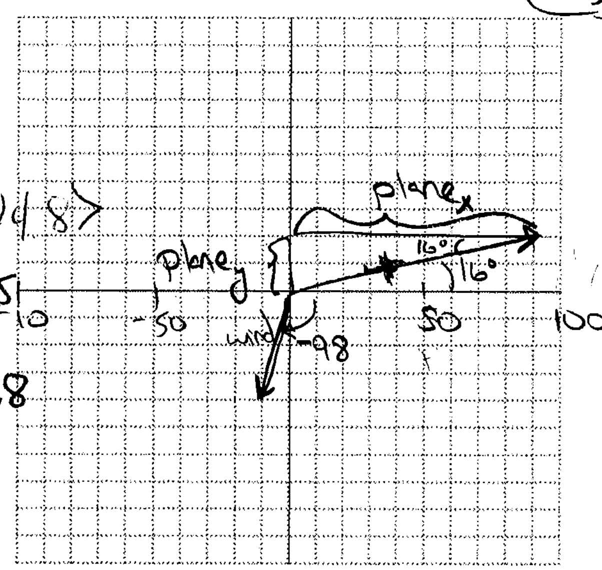
(a) [1] Identify east on the axis and sketch the velocity vector of the plane. (+.5) (+.5)

(b) [1] Find the components of the velocity vector of the plane. $\langle 86.5, 24.8 \rangle$

Sohcahtoa

$\cos 16° = \dfrac{\text{plane}_x}{90} \Rightarrow \text{plane}_x = 86.5$

$\sin 16° = \dfrac{\text{plane}_y}{90} \Rightarrow \text{plane}_y = 24.8$



(c) [3] The air is moving (wind) with the speed of 25 knots in the direction of −98° from due east. What is the plane's actual heading (direction)?

start (+.5)

notation/variable (+.5) The two velocity vectors add to get the actual heading

(+.5) $\overrightarrow{\text{wind velocity}} + \overrightarrow{\text{plane velocity}}$

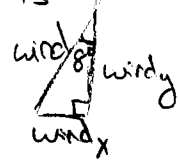
$\langle -3.5, -24.8 \rangle + \langle 86.5, 24.8 \rangle$

(+.5) $\langle 83, 0 \rangle$

actual heading is due East

$\overrightarrow{\text{wind vel}}$ components

(+1) $\sin 8° = \dfrac{\text{wind}_x}{\|\text{wind}\|}$



$\Rightarrow \text{wind}_x = 25 \sin 8°$

$= 3.5$ in neg. direction

$\cos 8° = \dfrac{\text{wind}_y}{\|\text{wind}\|}$

$\Rightarrow \text{wind}_y = 25 \cos 8°$

$= 24.8$ in neg direction

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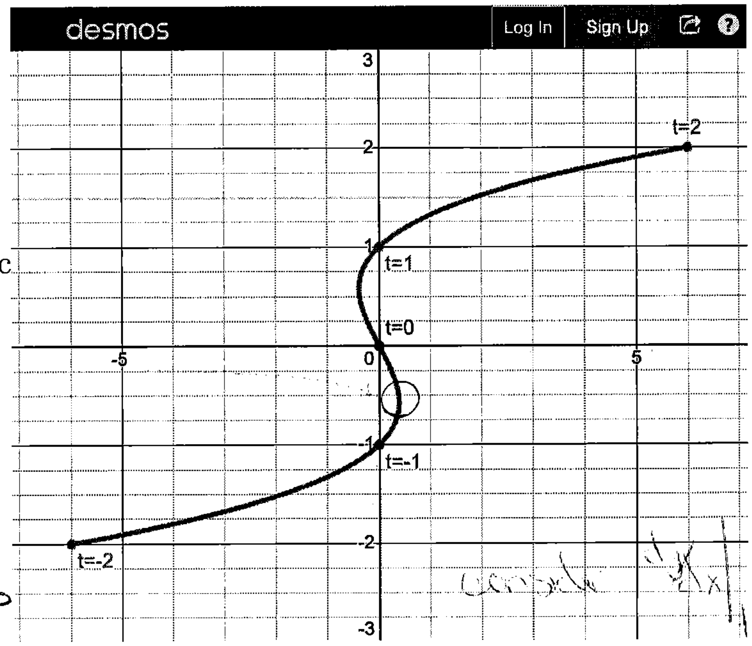
2
2
4

Median 68%

5. Consider the parametric curve $x = f(t)$, $y = g(t)$ where $-2 \le t \le 2$, graphed below for the following questions.



(a) [1] Identify/Estimate the point on the parametric curve when $t = -0.5$.

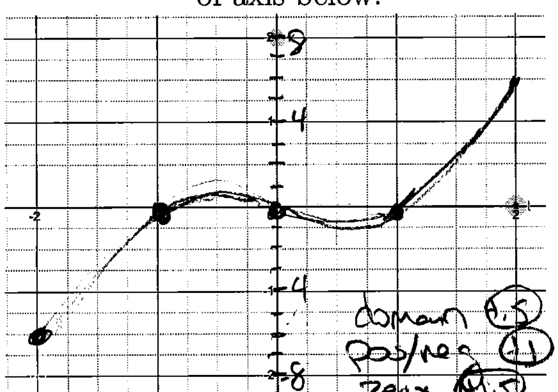
$\approx \left(\frac{1}{2}, -\frac{1}{2}\right)$

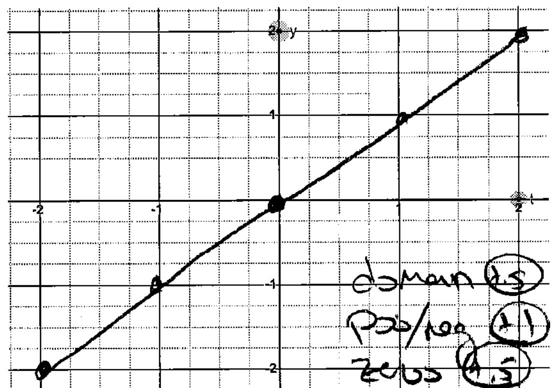
(b) [1] (PracticeExam1 #4) Identify/Estimate a point where $\frac{dy}{dx}$ is not defined.

places w/ vertical tangents $\left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)$

places w/ end points: $(-6,-2), (6,2)$

(c) [6] (WrittenHW§10.1#32) Sketch the equations $x = f(t)$ and $y = g(t)$ on the pair of axis below.





t	x	y
-2	-6 neg	-2
-1	0 pos	-1
0	0 neg	0
1	0 pos	1
2	6	2

(d) [4] (WebHW10.2#3) Given the following information, find the line tangent to the curve $x = f(t)$, $y = g(t)$ when $t = \frac{3}{2}$. Use whatever form of a line you like (eg. parametric, slope-intercept, standard, etc)

$f\left(\frac{3}{2}\right) = 1.9 \qquad g\left(\frac{3}{2}\right) = 1.5 \qquad \frac{df}{dt}\left(\frac{3}{2}\right) = 2.8 \qquad \frac{dg}{dt}\left(\frac{3}{2}\right) = 1$

looking for line in 2D: $y - y_0 = m(x - x_0)$ (+.5) OR

m = slope of line tangent to graph @ $t = \frac{3}{2}$

$= \left.\frac{dy}{dx}\right|_{t=3/2} = \frac{\left.(dy/dt)\right|_{t=3/2}}{\left.(dx/dt)\right|_{x=3/2}} = \frac{1}{2.8}$ (+1.5)

$\approx .35$

So $y - 1.5 = .35(x - 1.9)$

looking for line $\langle x, y\rangle = \langle x_0, y_0\rangle + t\vec{v}$ (+.5)

$\vec{v}$ = directional vector of curve @ $t = \frac{3}{2}$ (+.5)

$= \langle 2.8, 1\rangle$ (+1)

So $\langle x, y\rangle = \langle 1.9, 1.5\rangle + s\langle 2.8, 1\rangle$ as $s \in \mathbb{R}$

form (+.5)

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