A proof of the Brown–Goodearl Conjecture for module-finite weak Hopf algebras

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Hopf algebras

- Throughout, work over a field $\mathbb{k}$.
- A bialgebra $H$ over $\mathbb{k}$ is a $\mathbb{k}$-algebra $(H, m, u)$ and a $\mathbb{k}$-coalgebra $(H, \Delta, \varepsilon)$ such that

  (a) $m$ and $u$ are coalgebra morphisms

  $$\Delta(ab) = \Delta(a)\Delta(b) \quad \text{and} \quad \Delta(1) = 1 \otimes 1$$

  (a’) $\Delta$ and $\varepsilon$ are algebra morphisms

  $$\Delta(ab) = \Delta(a)\Delta(b) \quad \text{and} \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b).$$

- **Sweedler Notation.** $\Delta(a) = a_1 \otimes a_2$.
- A Hopf algebra is a bialgebra with a $\mathbb{k}$-linear antipode $S$

  $$S(a_1)a_2 = a_1S(a_2) = \varepsilon(a).$$
Hopf algebras

Example

$G$ a group. The group algebra $\mathbb{k}G$.

For $g \in G$: $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $S(g) = g^{-1}$

Example

$L$ a Lie algebra. The universal enveloping algebra $U(L)$.

For $x \in L$: $\Delta(x) = 1 \otimes x + x \otimes 1$, $\varepsilon(x) = 0$, $S(x) = -x$.

Example

$H$ a finite-dimensional Hopf algebra. Then the dual $H^* = \text{Hom}_{\mathbb{k}}(H, \mathbb{k})$ is also a Hopf algebra.
Hopf algebras are nice

- Hopf algebras possess good homological properties.
- If $\text{char } k = 0$, then every affine commutative Hopf algebra over $k$ is regular (finite global dimension).
- Every affine commutative Hopf algebra over $k$ is Gorenstein (finite injective dimension).
- Finite-dimensional Hopf algebras are Frobenius.
The Brown–Goodearl Conjecture

**Question [Brown 1998]**
Let $H$ be an affine noetherian Hopf algebra satisfying a polynomial identity. Does $H$ have finite injective dimension?

**Theorem [Wu–Zhang 2003]**
Yes!

**Question [Brown–Goodearl and Wu–Zhang]**
Does every noetherian Hopf algebra have finite injective dimension?
The Brown–Goodearl Conjecture

Stronger Question [Brown–Goodearl and Wu–Zhang]
Is every noetherian Hopf algebra Artin–Schelter Gorenstein?

Definition
A \( k \)-algebra \( A \) is Artin–Schelter Gorenstein of dimension \( d \) if:
(1) \( _AA \) has finite injective dimension \( d \),
(2) for all finite-dimensional \( A \)-modules \( _AM \),
\[
\operatorname{Ext}^i_A(M, A) = \begin{cases} 
0, & i \neq d \\
\text{finite-dimensional}, & i = d.
\end{cases}
\]
(3) The right-sided versions of (1) and (2) hold.
Why are Hopf algebras nice?

- Let $H$-$\text{mod}$ be the category of left $H$-modules.
- Let $H$-$\text{mod}_{\text{fd}}$ be the category of finite-dimensional ones.
- If $H$ is a Hopf algebra, for $M, N \in H$-$\text{mod}$, $M \otimes_k N \in H$-$\text{mod}$ where
  \[ h.(m \otimes n) = h_1.m \otimes h_2.n. \]
- Uses the coalgebra structure of $H$.
- This makes $H$-$\text{mod}$ a monoidal category.
- $1 = H \mathbb{1}_k$ where $H$ acts by $\varepsilon$.
- The above works for any bialgebra. The antipode $S$ makes $H$-$\text{mod}_{\text{fd}}$ rigid. Every $M \in H$-$\text{mod}_{\text{fd}}$ has a left dual.
Weak Hopf algebras

- A weak bialgebra $H$ over $k$ is a $k$-algebra $(H, m, u)$ and a $k$-coalgebra $(H, \Delta, \varepsilon)$ such that

  1. $\Delta(ab) = \Delta(a)\Delta(b)$,
  2. $(\Delta \otimes \text{Id}) \circ \Delta = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$,
  3. $\varepsilon(abc) = \varepsilon(ab_1)e(b_2c) = \varepsilon(ab_2)e(b_1c)$.

- Bialgebra if and only if $\Delta(1) = 1 \otimes 1$ if and only if $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$.

- A weak Hopf algebra is a weak bialgebra with antipode $S$:

  $S(a_1)a_2 = 1_1\varepsilon(a_12)$, \quad $a_1S(a_2) = \varepsilon(1_1a)1_2$, \quad $S(a_1)a_2S(a_3) = S(a)$. 
Why weak Hopf algebras?

• Introduced by [Böhm–Nill–Szlachanyi 1999], motivated by physics: study symmetries in conformal field theory.

• Axioms are self-dual, so the dual of a finite-dimensional weak Hopf algebra is again a weak Hopf algebra.

Example

If $H, K$ are bialgebras, then $H \oplus K$ is an algebra as usual and a coalgebra under

$$\Delta(h, k) = (h_1, 0) \otimes (h_2, 0) + (0, k_1) \otimes (0, k_2)$$

$$\varepsilon(h, k) = \varepsilon_H(h) + \varepsilon_K(k)$$

But $\Delta(1, 1) = (1, 0) \otimes (1, 0) + (0, 1) \otimes (0, 1) \neq (1, 1) \otimes (1, 1)$. So $H \oplus K$ not a bialgebra, only a weak bialgebra.
Why weak Hopf algebras?

- If \( G, H \) are groups, then \( G \sqcup H \) is not a group, but a groupoid.

**Example**

\( \mathcal{G} \) is a groupoid. \( k\mathcal{G} \) the groupoid algebra is a weak Hopf algebra.

For \( g \in \mathcal{G} \):

\[
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}.
\]

\[
\mathcal{G} = \begin{array}{c}
1 \\
\alpha \downarrow \\
\alpha^{-1} \uparrow \\
2
\end{array}
\]

Then \( 1 = e_1 + e_2 \) but \( \Delta(1) = e_1 \otimes e_1 + e_2 \otimes e_2 \neq 1 \otimes 1 \).

- For any (weak) Hopf algebra \( H \), the matrix algebra \( M_n(H) \) is a weak Hopf algebra.
Why weak Hopf algebras?

**Theorem [Hayashi 1999, Szlachányi 2001]**

Every *fusion category* is equivalent to $H\text{-mod}_{fd}$ for some weak Hopf algebra $H$.

- Hopf algebras **not general enough** to describe all fusion categories.
- If $(H, m, u, \Delta, \varepsilon)$ is an algebra and coalgebra such that $\Delta(ab) = \Delta(a)\Delta(b)$, then

  $$\Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$$

  $\Rightarrow$ comod-$H$ and $H$-comod are monoidal,

  $$\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$$

  $\Rightarrow$ mod-$H$ and $H$-mod are monoidal.

- (But not $\otimes_k$!) [Nill 1998], [Böhm–Caenepeel–Janssen 2011]
The Brown–Goodearl Question

- \( H \)-mod monoidal structure \( \Rightarrow H \) good homological properties.

**Question**

If \( H \) is a noetherian weak Hopf algebra, does \( H \) have finite injective dimension? Is \( H \) AS Gorenstein?

- **Remark.** The direct sum of AS Gorenstein algebras of different dimensions is not AS Gorenstein.

**Question**

If \( H \) is a noetherian weak Hopf algebra, does \( H \) have finite injective dimension? Is \( H \) a direct sum of AS Gorenstein algebras?
The Brown–Goodearl Question

Theorem [Rogalski, —, Zhang]

If $H$ is a weak Hopf algebra or quasi-Hopf algebra which is module-finite over its affine center, then $H$ is a direct sum of AS Gorenstein algebras. In particular, $H$ has finite injective dimension.
Thank you!