Ring and Module Theory Qual Review

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1 (Some) qual problems

- (Spring 2007, 2) Let I, J be two ideals in a commutative ring R (with unit).
  (a) Define $K = \{r : rJ \leq I\}$. Show that K is an ideal
  (b) If R is a PID, so $I = \langle i \rangle$, $J = \langle j \rangle$, give a formula for a generator k of K.

- (Spring 2007, 3) Describe up to isomorphism all the $\mathbb{R}[x]$-module structures one might put on a 3 dimensional real vector space (extending the $\mathbb{R}$ action).

  Fundamental theorem of modules over a PID. Choose your favorite canonical form. Note that $\mathbb{R}[x]$ *is infinite dimensional so an* $\mathbb{R}[x]$*-module must be a direct sum of* $\mathbb{R}[x]/(a_i)$ *(which has dimension* $\deg a_i$*). The dimensions must sum to three.*

- (Fall 2009, 8) Let R be a commutative ring with identity. Suppose I and J are ideals of R such that R/I and R/J are noetherian rings. Prove that $R/(I \cap J)$ is also a noetherian ring.

- (Spring 2012, 3) Let R be a commutative ring.
  (a) Suppose that R is noetherian. Show that if $\varphi : R \to R$ is a surjective ring homomorphism, then it is injective.
  Construct an ascending chain by taking nested kernels. Use ACC.
  (b) If R is not noetherian, must a surjective ring homomorphism be injective? Prove or give a counterexample.
  Noetherianness is some finiteness condition, so it stands to reason that part (a) is true. Part (b) can't possibly be, because rings can be big. You should know an example of a non-noetherian commutative ring, $F[x_1, x_2, \ldots]$*, the polynomial ring in countably many variables. Just send* x_i *to* x_{i-1} *and* x_1 *to* 0.

- (Spring 2012, 5) Consider $f \in F[x]$ where F is an algebraically closed field. Suppose that f has the property that for all matrices $A \in M_n(F)$ of any size n, if $f(A) = 0$, then A is a diagonalizable matrix; then we say the polynomial f *forces diagonalizability.*
  (a) Characterize a simple rule exactly which polynomials in $F[x]$ force diagonalizability. Prove your answer.
  (b) Fix $m \geq 1$. Is every square matrix A with entries in F satisfying $A^m = I$ diagonalizable? (The answer depends on F).

- (Fall 2009, 2) Let R denote a commutative ring and I an ideal, $I \neq R$.

(a) Give an example where R/I has nilpotents but R doesn't.

Take a domain and mod out by a power of some element. $\mathbb{Z}/4\mathbb{Z}$ *works.*

(b) Give an example where R has nilpotents but R/I doesn't.

For a general example, take any ring with nilpotents and let I *be the nilradical. It is easy to prove that* R/I *has no nilpotents. For an explicit example you can always take* $R = \mathbb{Z}/4\mathbb{Z}$ *and* $I = 2\mathbb{Z}/4\mathbb{Z}$.

- (Fall 2009, 3) Let $\varphi : \mathbb{C}[x] \to F$ be a ring homomorphism where F is a field, $\varphi(1) \neq 0$.

  (a) Give an example where φ is not onto.

  Just take a bigger field. $\mathbb{C}[x] \to \mathbb{C}(x)$ *works nicely.*

  (b) If φ is onto, show that $F \cong \mathbb{C}$.

  The kernel of φ *must be a maximal ideal. By the Nullstellensatz you know that these are all of the form* $(x - \alpha)$*, so the quotient is* $\mathbb{C}$.

- (Fall 2009, 4)

  (a) Give an example of two finitely generated $\mathbb{Z}$-modules, M and N such that M, N are not isomorphic (as $\mathbb{Z}$-modules) but $\mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q} \otimes_{\mathbb{Z}} N$ (as $\mathbb{Q}$-modules).

  You can take $M = \mathbb{Q}$ *and* $N = \mathbb{Z}$*. Then both are isomorphic to* $\mathbb{Q}$.

  (b) Let M be a finitely generated $\mathbb{R}[x]$-module, described using the classification of f.g. modules over a PID. Give a similar description of $\mathbb{C}[x] \otimes_{\mathbb{R}[x]} M$ as a $\mathbb{C}[x]$-module.

  Use elementary divisors. You can show using the universal property that $\mathbb{C}[x] \otimes_{\mathbb{R}[x]} \mathbb{R}[x]/(f) \cong \mathbb{C}[x]/(f)$*. The only thing that will change is the irreducibles in* $\mathbb{R}[x]$ *which do not remain irreducible in* $\mathbb{C}[x]$*, i.e., the quadratic irreducibles with two complex roots. These you can break up using CRT.*

2 (Some) ring things to know

- Basic facts and definitions (homomorphisms, isomorphism theorems, subrings, ideals, quotient rings, etc.)
- Any finite integral domain is a field.
- Isomorphism theorems
- A commutative ring R is a field if and only if its only ideals are 0 and R.
- Kernels of ring homomorphisms are ideals.
- For a commutative ring R and an ideal I, R/I is a domain (resp. field) if and only if I is prime (resp. maximal).
- (Chinese Remainder Theorem) Let $A_1, \ldots, A_k$ be ideals of R. Then map

$$R \to R/A_1 \times \cdots \times R/A_k$$

has kernel $A_1 \cap \cdots \cap A_k$. If for each $i \neq j$, A_i and A_j are comaximal, then the map is surjective and $A_1 \cap \cdots A_k = A_1 \cdots A_k$ so

$$R/(A_1 \cdots A_k) = R/(A_1 \cap \cdots \cap A_k) \cong R/A_1 \times \cdots \times R/A_k.$$

- fields $\subsetneq$ Euclidean domains $\subsetneq$ PIDs $\subsetneq$ UFDs $\subsetneq$ Integral domains $\subsetneq$ Rings
  Examples showing strict inclusion: $F[x]$, $\mathbb{Z}[(1+\sqrt{-19})/2]$, $F[x,y]$, $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}_4$
- A Euclidean domain is a PID, an ideal is generated by an element of minimum norm.
- If $(a,b) = (d)$, where $d = \gcd(a,b)$.
- If $R[x]$ is a PID then R is a field.
- If F is a field then $F[x]$ is a Euclidean domain.
- Every prime ideal in a PID is maximal.
- In a UFD, an element is prime if and only if it is irreducible.
- R is a UFD if and only if $R[x]$ is a UFD.
- Gaussian integers
- (Gauss' Lemma) Let R be a UFD with field of fractions F and $p(x) \in R[x]$. If $p(x)$ is reducible in $F[x]$ then $p(x)$ is reducible in $R[x]$.
- (Eisenstein's criterion) Let P be a prime ideal of an integral domain R and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_x + a_0$ be a polynomial in $R[x]$. Suppose $a_{n-1}, \ldots, a_1, a_0 \in P$ and $a_0 \notin P^2$. Then $f(x)$ is irreducible.

3 (Some) module things to know

- Basic facts and definitions (homomorphisms, isomorphism theorems, submodules, quotient ideals, direct products, cyclic modules)
- An $F[x]$-module V is an F-vector space with a linear transformation $V \to V$.
- Free modules (Let A be any set and consider the free module $F(A)$. If M is any R-module, $\varphi A \to M$ any homomorphism, this lifts to a unique R-module homomorphism $\Phi : F(A) \to M$.)
- Tensor product (commutative, associative, right exact, splits over direct sums)
- (Universal property) Any R-bilinear map $M \times N \to L$ induces a unique R-module homomorphism $M \otimes_R N \to L$.
- Extension of scalars
- Hom (is left exact)

- Exact sequences (short, split)
- Projective modules (direct summand of a free module, lifting property, every short exact sequence ending with a projective splits)

  (Lifting property). Given a surjection $M \to N$ and any homomorphism $\varphi' : P \to N$, φ lifts to a (not necessarily unique) homomorphism $P \to N$.
- Injective modules (lifting property, every short exact sequence starting with an injective splits)

  (Lifting property dual to the projective property). For any injection $L \to M$, and any homomorphism $\varphi L \to Q$, φ lifts to a homomorphism $Q \to M$.
- (Baer's criterion) An R-module Q is injective if and only if for every left ideal I, any module homomorphism $I \to Q$ can be extended to one $R \to Q$.
- If R is a PID, Q is injective if and only if $rQ = Q$ for all $0 \neq r \in R$.
- Flat modules (tensoring with a flat module is exact)
- M is a noetherian R-module if nad only if every nonempty set of submodules of M contains a maximal element if and only if every submodule of$M$ is finitely generated.
- (Fundamental Theorem of Modules over a PID)

  Invariant factor form

  Let R be a PID and M a finitely generated R-module. Then

  $$M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_m)$$

  for some $r \in \mathbb{Z}_{\geq 0}$ and $a_1 \mid a_2 \mid \cdots \mid a_m$.

  Rational canonical form

  Elementary divisor form:

  $$M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus \cdots R/(p_t^{\alpha_t})$$

  for some $r \in \mathbb{Z}_{\geq 0}$ and $p_1^{\alpha_1}, \ldots, p_t^{\alpha_t}$ are positive powers of not necessarily distinct primes.

  Jordan canonical form
- Characteristic polynomials, minimal polynomials