Algebra Practice Qual

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Do as many problems as you can, as completely as you can. If a problem has multiple parts, you may use the result of any part in the proof of another part of that problem. You have two hours.

1 Group theory

1. (Wisconsin, January 2012) Let G be a finite group of order $4312 = 2^3 \cdot 7^2 \cdot 11$.
   (a) Show that G has a subgroup of order 77.
   (b) Prove that G has a subgroup of order 7 whose normalizer in G has index dividing 8.
   (c) Conclude that G is not simple.

2. (UIUC, January 2013)
   (a) Show that every group of order 77 is abelian.
   (b) Show that every group of order $135 = 5 \cdot 3^3$ is nilpotent.

2 Ring and module theory

3. (Wisconsin, January 2004) Let k be a field and let R be the subring of the polynomial ring $k[x]$ given by all polynomials with x-coefficient equal to 0.
   (a) Prove that the elements x^2 and x^3 are irreducible but not prime in the ring R.
   (b) Show that R is a noetherian ring, and that the ideal I of R consisting of all polynomials in R with constant term 0 is not principle.

4. (UCSD, May 2006) Let R be a commutative ring and let M and N be R-modules. Prove the following
   (a) If M and N are projective than $M \otimes_R N$ is projective.
   (b) If M and N are flat then $M \otimes_R N$ is flat
   (c) If M is flat and N is injective that $\mathrm{Hom}_R(M, N)$ is injective.

5. (Maryland, January 2008) Let $n \geq 1$ and let $f(x), g(x) \in \mathbb{C}[x]$ be monic polynomials. Assume that $\deg g(x) = n$, that $f(x)$ divides $g(x)$ and that every root of g is a root of f. Show that there is a linear transformation of $\mathbb{C}^n$ with minimal polynomial $f(x)$ and characteristic polynomial $g(x)$.

3 Field theory

6. (UIUC, January 2014)

   (a) Let E/F be a Galois extension of degree p^k. Prove that there exists an intermediate field K with $[E : K] = p$ and K/F Galois of degree p^{k-1}.

   (b) Determine the splitting field extension $E/\mathbb{Q}$ of the polynomial $x^4 - 2 \in \mathbb{Q}[x]$.

   (c) Give an intermediate field K with $[E : K] = 2$ and $K/\mathbb{Q}$ Galois.

7. (Maryland, January 2011) Let F be a field and suppose that the additive group of F is a finitely generated abelian group. Show that F is a finite field.