High-Resolution Finite Volume Methods with Application to Volcano and Tsunami Modeling

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Outline

- Volcanic flows, ash plumes, pyroclastic flow
- Tsunami modeling, shallow water equations

- Finite volume methods for hyperbolic equations
- Conservation laws and source terms
- Riemann problems and Godunov’s method
- Wave propagation form
- Wave limiters and high-resolution methods
- Software: CLAWPACK
Some collaborators

Algorithms, software
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Donna Calhoun, UW
Phil Colella, UC-Berkeley
Jan Olav Langseth, Oslo
Sorin Mitran, UNC
James Rossmanith, Michigan
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Tsunamis
David George, UW
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Volcanos
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Roger Denlinger, USGS CVO
Dick Iverson, USGS CVO
Alberto Neri, Pisa
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Marica Pelanti, Donna Calhoun, Joe Dufek, and David George

at Mount St. Helens
Volcanic flows

- Flow of magma in conduit
- Little dissolved gas $\implies$ lava flows
- Dissolved gas expansion $\implies$ phase transition, ash jet
- Ash plumes, Plinian columns
- Collapsing columns, pyroclastic flows or surges
- Lahars (mud flows)
- Debris flows
Volcanic Ash Plumes
Pyroclastic Flows
I. Pyroclastic dispersion dynamics of pressure-balanced eruptions

Influence of the diameter $D_v$ and the exit velocity $v_v$

Regions of different types of eruption columns (Neri–Dobran, 1994).

Characteristic features of a collapsing column.
Pyroclastic dispersion dynamics

Vent conditions and physical properties [Neri–Dobran, 1994]:

<table>
<thead>
<tr>
<th>$p_v$ [MPa]</th>
<th>$T_v$ [K]</th>
<th>$\alpha_{dv}$</th>
<th>$d$ [µm]</th>
<th>$\rho_d$ [kg/m$^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1200</td>
<td>0.01</td>
<td>10</td>
<td>2300</td>
</tr>
</tbody>
</table>

Gas and dust in thermal and mechanical equilibrium at the vent.

Test 1. $D_v = 100$ m, $v_v = 80$ m/s. → Collapsing volcanic column

Test 2. $D_v = 100$ m, $v_v = 200$ m/s. → Transitional/Plinian column
Numerical Experiments

Injection of a hot supersonic particle-laden gas from a volcanic vent into a cooler atmosphere.

- Initially: Standard atmosphere vertically stratified in pressure and temperature all over the domain;
- At the vent: Gas pressure, velocities, temperatures, volumetric fractions of gas and dust assumed to be fixed and constant;
- Ground boundary: modeled as a free-slip reflector;
- Other boundaries:
  - **2D experiments**: Axisymmetric configuration. Symmetry axis: free-slip reflector; Upper and right-hand edges of the domain: free flow boundaries (all the variables gradients set to zero).
  - **Fully 3D experiments**: Upper and lateral sides: free-flow boundaries.
$D_v = 100 \text{ m}, v_v = 80 \text{ m/s}. \text{Collapsing column.}$

Dust density at $t = 10, 30, 35, 70 \text{ s}$. Uniform grid, $200 \times 100$ cells. Cell size = 10 m. CFL = 0.9.
Physical Model

Two-phase fluid flow composed of solid particles (dust) in a gas.

Gas phase: compressible;
Dust phase: incompressible (constant microscopic mass density $\rho_d$).

Dust particles are assumed to be dispersed (vol. fraction $\alpha_d \ll 1$), with negligible particle-particle interaction. The solid phase is thus considered pressureless.

Model accounts for:

- Gravity;
- Interphase drag force;
- Interphase heat transfer.

Some of the neglected phenomena: viscous stress, turbulence.
Model Equations

Conservation of mass, momentum, and energy for gas and dust

\[ \rho_t + \nabla \cdot (\rho \mathbf{u}_g) = 0, \]

\[ (\rho \mathbf{u}_g)_t + \nabla \cdot (\rho \mathbf{u}_g \otimes \mathbf{u}_g + p \mathbf{I}) = \rho \mathbf{g} - D(\mathbf{u}_g - \mathbf{u}_d), \]

\[ E_t + \nabla \cdot ((E + p) \mathbf{u}_g) = \rho \mathbf{u}_g \cdot \mathbf{g} - D(\mathbf{u}_g - \mathbf{u}_d) \cdot \mathbf{u}_d - Q(T_g - T_d), \]

\[ \beta_t + \nabla \cdot (\beta \mathbf{u}_d) = 0, \]

\[ (\beta \mathbf{u}_d)_t + \nabla \cdot (\beta \mathbf{u}_d \otimes \mathbf{u}_d) = \beta \mathbf{g} + D(\mathbf{u}_g - \mathbf{u}_d), \]

\[ \Omega_t + \nabla \cdot (\Omega \mathbf{u}_d) = \beta \mathbf{u}_d \cdot \mathbf{g} + D(\mathbf{u}_g - \mathbf{u}_d) \cdot \mathbf{u}_d + Q(T_g - T_d). \]

\[ \alpha_g, \alpha_d = \text{volume fractions (} \alpha_g + \alpha_d = 1, \alpha_d \ll 1); \]
\[ \rho_g, \rho_d = \text{material mass densities (} \rho_d = \text{const.); } \rho = \alpha_g \rho_g, \beta = \alpha_d \rho_d = \text{macroscopic densities}; \]
\[ \mathbf{u}_g, \mathbf{u}_d = \text{velocities; } p_g = \text{gas pressure, } p = \alpha_g p_g; \]
\[ e_g, e_d = \text{specific total energies, } E = \alpha_g \rho_g e_g, \Omega = \alpha_d \rho_d e_d; \]
\[ e_g = \epsilon_g + \frac{1}{2}||\mathbf{u}_g||^2, e_d = \epsilon_d + \frac{1}{2}||\mathbf{u}_d||^2; \epsilon_g, \epsilon_d = \text{specific internal energies; } T_g, T_d = \text{temperatures; } \]
\[ \mathbf{g} = (0, 0, -g) = \text{gravity acceleration (z direction), } g = 9.8 \text{ m/s}^2; \]
\[ D = \text{drag function; } Q = \text{heat transfer function.} \]
Closure Relations

Gas equation of state: \[ p_g = (\gamma - 1)\rho_g\epsilon_g, \quad \gamma = c_{pg}/c_{vg} = \text{const.}; \]

Dust energy relation: \[ \epsilon_d = c_{vd}T_d, \quad c_{vd} = \text{const.}; \]

Drag

\[ D = \frac{3}{4}C_d \frac{\beta \rho}{\rho_d d} \| \mathbf{u}_g - \mathbf{u}_d \|, \]

\( d = \) dust particle diameter, \( C_d = \) drag coefficient,

\[ C_d = \begin{cases} \frac{24}{Re} \left( 1 + 0.15Re^{0.687} \right) & \text{if } Re < 1000, \\ 0.44 & \text{if } Re \geq 1000, \end{cases} \]

\( Re = \) Reynolds number = \( \frac{\rho d \| \mathbf{u}_g - \mathbf{u}_d \|}{\mu} \), \( \mu = \) dynamic viscosity of the gas.

Heat transfer

\[ Q = \frac{Nu 6\kappa_g \beta}{\rho_d d^2}, \]

\( Nu = \) Nusselt number = \( 2 + 0.65Re^{1/2}Pr^{1/3} \), \( Pr = \) Prandtl number = \( \frac{c_{pg} \mu}{\kappa_g} \), \( \kappa_g = \) gas thermal conductivity.
Shock structure in a supersonic jet

Illustration of an overpressured jet (JANNAF, 1975).
Overpressured jet: Mach number and normal Mach number at $t = 30 \, \text{s}$.

No crater $\rightarrow$

Crater 30° $\rightarrow$

Normal Mach number of the mixture (to highlight normal discontinuities)

$$M_m = \frac{u_m \cdot \nabla p_g}{c_m ||\nabla p_g||},$$

$$c_m^2 = \frac{\rho_g c_g^2}{\alpha_g \rho_m}, \quad c_g = \sqrt{RT_g},$$

$$u_m = \frac{\alpha_g \rho_g u_g + \alpha_d \rho_d u_d}{\rho_m},$$

$$\rho_m = \alpha_g \rho_g + \alpha_d \rho_d.$$
Comparison: CLAWPACK vs. PDAC2D (Neri–Ongaro, INGV, Pisa, Italy).

No crater

Crater 30°

Dust density at \( t = 10 \) and 20 s.
Mount St. Helens
Blast zone at Mount St. Helens
Trees blown down by MSH blast

http://volcanoes.usgs.gov/Hazards/Effects/MSHsurge_effects.html
Mount St. Helens
High-pressure initial blast
AMR computation
Volcanic Debris Flow
Volcanic Debris Flow

Mount St. Helens May 18, 1980 Devastation

Outline of crater
Pyroclastic flow deposits
Mudflow deposits
Lateral blast deposits
Debris avalanche deposits

Test flume studies

Cascade Volcano Observatory (CVO), Vancouver, Washington

http://vulcan.wr.usgs.gov/
Sand flume with topography

Recent results of Dick Iverson and Roger Denlinger, CVO

Experiments on small-scale sand flume with topography.

Compared to predictions from shallow-flow Savage-Hutter type model for granular avalanches.

Coulomb friction for shear and normal stresses on internal and bounding surfaces.

Finite-volume wave propagation method using finite element computation of stresses in Riemann solver.

Flow over steep topography.
Sand flume with topography
Sand on a flume with topography
Tsunamis

Generated by

- Earthquakes,
- Landslides,
- Submarine landslides,
- Volcanos,
- Meteorite or asteroid impact
Tsunamis

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- Small amplitude in ocean ($< 1$ meter) but can grow to 10s of meters at shore.
- Run-up along shore can inundate 100s of meters inland
- Long wavelength (as much as 200 km)
- Propagation speed $\sqrt{gh}$
- Average depth of Pacific is 4km $\implies$ average speed 200 m/s
1993 Okushiri tsunami

http://www.pmel.noaa.gov/tsunami/aerial_photo_okushiri.html
Catalina Workshop — June, 2004

3rd Int’l workshop on long-wave runup models

Benchmark Problem 2:

Fig. 1 Offshore Profile
Shallow water equations with topography $B(x, y)$

$$h_t + (hu)_x + (hv)_y = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x(x, y)$$

$$(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghB_y(x, y)$$

Applications:

- Tsunamis
- Estuaries
- River flooding, dam breaks
- Debris flows from volcanic eruptions
Frame 56
Channel 5

Surface Elevation at Channel 5 (4.521,1.196)

- Computed Solution
- Experimental Solution
Hyperbolic Partial Differential Equations

Model advective transport or wave propagation

Advection equation:
\[ q_t + uq_x = 0, \quad q_t + uq_x + vq_y = 0 \]

First-order system:
\[ q_t + Aq_x = 0, \quad q_t + Aq_x + Bq_y = 0 \]

where \( q \in \mathbb{R}^m \) and \( A, B \in \mathbb{R}^{m \times m} \).

Hyperbolic if

1D: \( A \) is diagonalizable with real eigenvalues,
2D: \( \cos(\theta)A + \sin(\theta)B \) is diagonalizable with real eigenvalues, for all angles \( \theta \).

Eigenvalues give wave speeds, eigenvectors the wave forms.
Nonlinear conservation laws

\[ q_t + f(q)_x = 0, \text{ where } f(q) \text{ is the flux function.} \]

Quasi-linear form: \[ q_t + f'(q)q_x = 0. \]

Hyperbolic if \( f'(q) \) is diagonalizable with real eigenvalues.

Eigenvalues depend on solution

\[ \implies \text{ characteristics may converge.} \]

\[ \implies \text{ Shock formation and discontinuous solutions.} \]
Finite-difference Methods

- Pointwise values \( Q^n_i \approx q(x_i, t_n) \)
- Approximate derivatives by finite differences
- Assumes smoothness

Finite-volume Methods

- Approximate cell averages: \( Q^n_i \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n) \, dx \)

- Integral form of conservation law,

\[
\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t) \, dx = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))
\]

leads to conservation law \( q_t + f_x = 0 \) but also directly to numerical method.
Finite volume method

\[ Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n) \, dx \]

Integral form:
\[ \frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t) \, dx = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t)) \]

Numerical method:
\[ Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} (F_{i+1/2}^n - F_{i-1/2}^n) \]

Numerical flux:
\[ F_{i-1/2}^n \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}, t)) \, dt. \]
The Riemann problem

The Riemann problem for $q_t + f(q)_x = 0$ has special initial data

$$q(x, 0) = \begin{cases} q_l & \text{if } x < x_{i-1/2} \\ q_r & \text{if } x > x_{i-1/2} \end{cases}$$

Dam break problem for shallow water equations
The Riemann problem

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Dam break problem for shallow water equations
Riemann solution for the SW equations

rarefaction wave

\[
\begin{bmatrix}
h_l \\
u_l
\end{bmatrix}
\]

contact

\[
\begin{bmatrix}
h^* \\
u^*
\end{bmatrix}
\]

shock

\[
\begin{bmatrix}
h_r \\
u_r
\end{bmatrix}
\]
Godunov’s method

$Q^n_i$ defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q^n_i \quad \text{for} \quad x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\implies$ Riemann problems.

\[ F^n_{i-1/2} = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q^\downarrow(Q^n_{i-1}, Q^n_i)) \, dt = f(q^\downarrow(Q^n_{i-1}, Q^n_i)). \]
The Roe solver uses the solution to a linear system

\[ q_t + \hat{A}_{i-1/2} q_x = 0, \quad \hat{A}_{i-1/2} = f'(q_{\text{ave}}). \]

All waves are simply discontinuities.

Typically a fine approximation if jumps are approximately correct.
Wave decomposition for shallow water

\[ q = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \end{bmatrix} \]

Jacobian: \[ f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix} \]

Eigenvalues: \[ \lambda^1 = u - \sqrt{gh}, \quad \lambda^2 = u + \sqrt{gh}, \]

Eigenvectors: \[ r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}, \quad r^2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix} \]

Wave decomposition:

\[ Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}. \]
Challenges for tsunami modeling

Want robust method with high resolution corrections that “captures” moving shoreline location

Need robust dry state Riemann solver

   Modified HLLE solver that avoids negative $h$

Bottom bathymetry / topography

   Source term incorporated into Riemann solver

f-wave formulation for $q_t + f(q)_x = \psi(q)$:

   Split $f(Q_i) - f(Q_{i-1}) - \Delta x \Psi_{i-1/2} = \sum_p \beta^p_{i-1/2} r^p_{i-1/2}$
Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $\mathcal{W}^p$ propagating at constant speed $s^p$.

\[
Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.
\]

\[
Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[ s^2 \mathcal{W}^2_{i-1/2} + s^3 \mathcal{W}^3_{i-1/2} + s^1 \mathcal{W}^1_{i+1/2} \right].
\]
Upwind wave-propagation algorithm

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left[ \sum_{p=1}^{m} (s_{i-1/2}^p)^+ \mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m} (s_{i+1/2}^p)^- \mathcal{W}_{i+1/2}^p \right] \]

where

\[ s^+ = \max(s, 0), \quad s^- = \min(s, 0). \]

Note: Requires only waves and speeds.

Applicable also to hyperbolic problems not in conservation form.

Conservative if waves chosen properly, e.g. using Roe-average of Jacobians.

Great for general software, but only first-order accurate (upwind).
Wave-propagation form of high-resolution method

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left[ \sum_{p=1}^{m} (s_{i-1/2}^{p})^+ \mathcal{W}_{i-1/2}^{p} + \sum_{p=1}^{m} (s_{i+1/2}^{p})^- \mathcal{W}_{i+1/2}^{p} \right] \]

\[ - \frac{\Delta t}{\Delta x} (\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}) \]

Correction flux:

\[ \tilde{F}_{i-1/2} = \frac{1}{2} \sum_{p=1}^{M_w} |s_{i-1/2}^{p}| \left( 1 - \frac{\Delta t}{\Delta x} |s_{i-1/2}^{p}| \right) \tilde{\mathcal{W}}_{i-1/2}^{p} \]

where \( \tilde{\mathcal{W}}_{i-1/2}^{p} \) is a limited version of \( \mathcal{W}_{i-1/2}^{p} \).
CLAWPACK

http://www.amath.washington.edu/~claw/

- Fortran codes with Matlab graphics routines.
- Many examples and applications to run or modify.
- 1d, 2d, and 3d.
- Adaptive mesh refinement.

User supplies:

- Riemann solver, splitting data into waves and speeds (Need not be in conservation form)
- Boundary condition routine to extend data to ghost cells
  Standard bc1.f routine includes many standard BC’s
- Initial conditions — qinit.f
Adaptive Mesh Refinement (AMR)

- Berger / Oliger / Colella
- Flag cells needing refinement
- Cluster into rectangular patches
- Refine in time also on patches
- Software:
  - AMRCLAW (Berger, RJL)
  - CHOMBO (Colella, et.al.)
  - CHOMBO-CLAW (Calhoun)
  - BEARCLAW (Mitran)
  - AMROC (Deiterding)
Some other applications

- Acoustics, ultrasound, seismology
- Elasticity, plasticity, soil liquifaction
- Flow in porous media, groundwater contamination
- Oil reservoir simulation
- Geophysical fbw on the sphere
- Chemotaxis and pattern formation
- Multi-fluid, multi-phase fbws, bubbly fbw
- Streamfunction–vorticity form of incompressible fbw
- Projection methods for incompressible fbw
- Combustion, detonation waves
- Astrophysics: binary stars, planetary nebulae, jets
- Magnetohydrodynamics, plasmas
- Relativistic fbw, black hole accretion
- Numerical relativity — gravitational waves, cosmology
Summary and extensions

- Applications to geophysical flows
- Scientific enquiry and hazard mitigation
- General formulation of high-resolution finite volume methods
- Applies to general conservation laws and nonconservative hyperbolic problems
- F-wave formulation for spatially varying fluxes and source terms
- Multi-dimensional extensions
- Adaptive mesh refinement
- CLAWPACK Software:
  http://www.amath.washington.edu/~claw