Immersed Interface Methods for Fluid Dynamics Problems

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Incompressible Navier-Stokes

\[ u_t + (u \cdot \nabla)u + \nabla p = \mu \nabla^2 u + f \]
\[ \nabla \cdot u = 0 \]

An immersed elastic membrane \( \Gamma \) exerts a singular force on the fluid,

\[ f(x, y) = \int_{\Gamma} F(s) \delta(x - X(s)) \delta(y - Y(s)) \, ds, \]

and moves with the fluid.
Peskin’s Heart Model

Originally developed to model blood flow in a beating heart and the operation of artificial heart valves.
Balloon in a driven cavity
Balloon in a driven cavity

\[ t = 0 \]

\[ t = 8 \]

\[ t = 14 \]

\[ t = 30 \]
Peskin’s Immersed Boundary Method

- Membrane represented by discrete control points $X^n_k$.
- Calculate force strength $F^n_k$ at each control point.
- Use discrete delta function to spread forces to nearby Cartesian grid points, yielding nonzero $f_{ij}$ at points near the interface.
- Advance the fluid equations on the uniform grid.
- Interpolate resulting velocity field $u_{ij}^{n+1}$ to control points to obtain $U^{n+1}_k$.
- Move control points by $X^{n+1}_k = X^n_k + \Delta t U^{n+1}_k$.
- Implicit or semi-implicit approach may be needed for stability.
Discrete delta function in 1D

Example: Hat function

Singular force \( F\delta(x - \alpha) \approx Fd_h(x_i - \alpha) \) on the grid. This is nonzero at only two points \( (x_j < \alpha < x_{j+1}) \):
Spring model of forces

The force $\vec{F}_k$ at $\vec{X}_k$ is computed based on the shape of the boundary.

**Example:** Spring model

$$\vec{F}_k = \sigma_{k+1/2}(\vec{X}_{k+1} - \vec{X}_k) - \sigma_{k-1/2}(\vec{X}_k - \vec{X}_{k-1}).$$
Spring model of forces

The force $\vec{F}_k$ at $\vec{X}_k$ is computed based on the shape of the boundary.

**Example:** Spring model

$$\vec{F}_k = \sigma_{k+1/2}(\vec{X}_{k+1} - \vec{X}_k) - \sigma_{k-1/2}(\vec{X}_k - \vec{X}_{k-1}).$$

For $\vec{X}(s)$ parameterized by unstretched length,

$$f(s, t) = \frac{\partial}{\partial s} (T(s, t)\tau(s, t)),$$

where

$$T(s, t) = T_0 \left( \left| \frac{\partial X(s, t)}{\partial s} \right| - 1 \right).$$
Jump conditions

With mass density $m(s)$:

$$m(s)X_{tt}(s,t) = f(s,t) - [p]n + \mu \left[ \frac{\partial u}{\partial n} \right].$$

Massless membrane: $m(s) = 0$

$$f = \text{elastic force (computed from } X^n\text{)}$$
$$f = f_n n + f_\tau \tau$$

$$[p] = f_n$$

$$\mu \left[ \frac{\partial u}{\partial n} \right] = -f_\tau \tau$$
Projection Method (one form)

\[
\begin{align*}
  u_t + (u \cdot \nabla) u + \nabla p &= \mu \nabla^2 u + f \\
  \nabla \cdot u &= 0
\end{align*}
\]

1. \( U^n \rightarrow U^* \) by solving

\[
  u_t + (u \cdot \nabla) u = \mu \nabla^2 u + f
\]

2. \( U^* \rightarrow U^{n+1} \) by solving

\[
  u_t + \nabla p = 0
\]

and requiring \( \nabla \cdot U^{n+1} = 0 \):

\[
\frac{U^{n+1} - U^*}{\Delta t} + \nabla p = 0
\]

\[\implies \Delta t \nabla^2 p = \nabla \cdot U^* \]
1. $U^n \rightarrow U^*$ by solving $u_t + (u \cdot \nabla)u = \mu \nabla^2 u + f$

2. $U^* \rightarrow U^{n+1}$ by solving $\Delta t \nabla^2 p = \nabla \cdot U^*$

**True solution:**
- $p$ should be discontinuous across $\Gamma$
- $u$ should be continuous but not smooth

**Numerical solution:**
- Singular source in $U^*$ leads to “delta function” in $U^*$
- $\nabla \cdot U^*$ gives “dipole source” for $\nabla^2 p$
- Results in “discontinuity” in $p$. 
**Immersed Interface Approach**

\[
\begin{align*}
 u_t + (u \cdot \nabla)u + \nabla p &= \mu \nabla^2 u + f \\
\nabla \cdot u &= 0
\end{align*}
\]

1. \( U^n \rightarrow U^* \) by solving

\[
 u_t + (u \cdot \nabla)u = \mu \nabla^2 u + f_r
\]

2. \( U^* \rightarrow U^{n+1} \) by solving

\[
 u_t + \nabla p = f_n
\]

and requiring \( \nabla \cdot U^{n+1} = 0 \):

\[
 \frac{U^{n+1} - U^*}{\Delta t} + \nabla p = f_n
\]

\[
 \Rightarrow \quad \Delta t \nabla^2 p = \nabla \cdot U^* + \Delta t \nabla \cdot f_n
\]
\[ U^n \rightarrow U^* \] by solving \( u_t + (u \cdot \nabla)u = \mu \nabla^2 u + f_r \)

\[ U^* \rightarrow U^{n+1} \] by solving \( \Delta t \nabla^2 p = \nabla \cdot U^* + \Delta t \nabla \cdot f_n \)

**Numerical solution:**

- Singular source in 1 is tangential to interface, so \( u \) remains bounded
- \( \nabla \cdot f_n \) gives correct dipole source for \( \nabla^2 p \)
- Use jump conditions on \( p \) and \( \partial p / \partial n \) while solving

\[
\Delta t \nabla^2 p = \nabla \cdot U^*
\]

using an immersed interface method.
Simple 1D example: $p_{xx} = f(x)$ with boundary conditions and jump condition $[p] = c$ at $x = \alpha$.

Or: $p_{xx} = f(x) + c\delta'(x - \alpha)$.

Want to set $p_{xx}(x_i) = f(x_i)$.

\[
p(x_{i-1}) = p(x_i) - hp_x(x_i) + \frac{1}{2}h^2 p_{xx}(x_i) - \cdots
\]

\[
p(x_{i+1}) = p(x_i) + hp_x(x_i) + \frac{1}{2}h^2 p_{xx}(x_i) + \cdots
\]

\[
\implies p_{xx}(x_i) \approx \frac{p(x_{i-1}) - 2p(x_i) + p(x_{i+1})}{h^2}
\]

So that \[\frac{p_{i-1} - 2p_i + p_{i+1}}{h^2} = f_i\]
\[ p_{xx} = f(x) + c\delta'(x - \alpha). \]

Want to set \( p_{xx}(x_j) = f(x_j) \).

\[ p(x_{j-1}) = p(x_j) - hp_x(x_j) + \frac{1}{2}h^2 p_{xx}(x_j) - \cdots \]
\[ p_{xx} = f(x) + c\delta'(x - \alpha). \]

Want to set \( p_{xx}(x_j) = f(x_j) \).

\[
p(x_{j-1}) = p(x_j) - hp_x(x_j) + \frac{1}{2}h^2 p_{xx}(x_j) - \cdots \\
p(x_{j+1}) = (p(x_j) + hp_x(x_j) + \frac{1}{2}h^2 p_{xx}(x_j) + \cdots) + c
\]

\[ \implies p_{xx}(x_j) \approx \frac{p(x_{j-1}) - 2p(x_j) + p(x_{j+1})}{h^2} - \frac{c}{h^2} \]

So that

\[ \frac{p_{j-1} - 2p_j + p_{j+1}}{h^2} = f_j + \frac{c}{h^2} \]

and similarly

\[ \frac{p_j - 2p_{j+1} + p_{j+2}}{h^2} = f_{j+1} - \frac{c}{h^2} \]
Example: \( u_{xx} + u_{yy} = \int_{\Gamma} \delta(x - X(s)) \delta(y - Y(s)) \, ds \)

We use the Dirichlet boundary condition which is determined from the exact solution

\[
u(x, y) = \begin{cases} 
1 & \text{if } r \leq 0.5 \\
1 + \log(2r) & \text{if } r > 0.5
\end{cases}
\]

where \( r = \sqrt{x^2 + y^2} \).

Using \( d_h(x) \):  

Using jump conditions:
Example with discontinuity in solution

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<th>$n$</th>
<th>global error</th>
<th>ratio</th>
<th>local error</th>
<th>ratio</th>
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<td>$1.93917 \times 10^{-3}$</td>
<td>1.9982</td>
</tr>
</tbody>
</table>
**Immersed Interface Method**

- Represent interface by spline through smaller number of control points:

\[
f(s, t) = \frac{\partial}{\partial s} (T(s, t)\tau(s, t)),
\]

where

\[
T(s, t) = T_0 \left( \left| \frac{\partial X(s, t)}{\partial s} \right| - 1 \right),
\]

- Can compute force and hence pressure jump at any point on membrane.

- Incorporate jump conditions into Taylor series expansion to derive finite-difference method that is pointwise second-order accurate.
Hybrid Finite-Volume / Finite-Difference Method

- Fluid equations solved on finite-volume grid.
- Cell-centered velocity advected using high-resolution methods (CLAWPACK)
- Edge velocities needed in advection algorithm are obtained by averaging cell-centered values.

\[
\begin{align*}
U_{i-1,j} & \quad V_{i-1,j} & \quad p_{i-1,j} \\
U_{i,j} & \quad V_{i,j} & \quad p_{i,j} \\
U_{i+1,j} & \quad V_{i+1,j} & \quad p_{i,j+1}
\end{align*}
\]
Hybrid Finite-Volume / Finite-Difference Method

- Divergence-free condition applied to the edge velocities.
- Finite-difference immersed interface method used to compute pressure at cell centers.
- Pressure correction is then applied to cell-centered velocities.
### Explicit Method

Given $X^k_n$, $U^n$, $u^n$ at start of time step.

**Step 1.** Solve $U_t + (u^n \cdot \nabla)U = 0$. Takes $U^n \rightarrow U^\dagger$.

**Step 2.** Solve $U_t = \mu \nabla^2 U + f_T$. Takes $U^\dagger \rightarrow U^*$.

**Step 3.** Average $U^*$ from adjacent grid cells to obtain $u^*$ at edges.

**Step 4.** Solve $\Delta t \nabla^2 p^{n+1} = \nabla \cdot u^*$ with $[p^{n+1}] = f_n$ to obtain $p^{n+1}$.

**Step 5.** Update $u^*$ based on $p^{n+1}$ to get $u^{n+1}$ (div free).

**Step 6.** Update $U^*$ based on $p^{n+1}$ to get $U^{n+1}$.

**Step 7.** Interpolate $U^{n+1}$ to marker points $X^n$ to obtain $U^{n+1}(X^n)$. Move membrane using

$$X^{n+1} = X^n + \Delta t U^{n+1}(X^n).$$
Trapezoidal Method

\[ U^n(X^n) = \text{velocities at marker points } X^n, \text{ interpolated from } U^n, \]
\[ U^{n+1}(X^{n+1}) = \text{velocities at marker points } X^{n+1}, \text{ interpolated from } \]
\[ U^{n+1}, \text{ determined by stepping forward from } U^n \text{ with forces} \]
\[ \frac{1}{2}(f(X^n) + f(X^{n+1})). \]

Would like:

\[ X^{n+1} = X^n + \frac{1}{2}\Delta t (U^n(X^n) + U^{n+1}(X^{n+1})). \]

Problem: Implicit in \( X^{n+1} \) and computing \( U^{n+1}(X^{n+1}) \) requires solving fluid equations and Poisson problem.
Implicit Method

Given $X^n_k, U^n, u^n$ at start of time step. 
Apply Quasi-Newton method to solve for $X^{n+1}$.

**Step I1.** Apply Step 1, the advection step. $U^n \rightarrow U^\dagger$.

**Step I2.** Make a guess $X^{[0]}$ for $X^{n+1}$ and set $I = 0$.

**Step I3.** Perform Steps 2–6 but replacing $f(X^n)$ by $\frac{1}{2}(f(X^n) + f(X^{[I]}))$ 
This gives provisional velocity field $U^{n+1}$.

**Step I4.** Evaluate 

$$g(X^{[I]}) = X^{[I]} - X^n - \frac{1}{2} \Delta t \left( U^n(X^n) + U^{n+1}(X^{[I]}) \right)$$

**Step I5.** Convergence check: If $\|g(X^{[I]})\| \leq \epsilon$ then set $X^{n+1} = X^{[I]}$, done.

Otherwise, update $X^{[I]}$ to $X^{[I+1]}$, set $I = I + 1$, and go to step I3.
Oscillating balloon

- Initial
- Resting
- Equilibrium
Volume Conservation and Accuracy

- Graphs showing radii in horizontal or vertical directions over time.
- Comparison of 256 x 256 (IIM) and 256 x 256 (IB).
- Plots of relative error with grid size.

Grid size vs. relative error:
- IIM
- IBM
  - 2nd–order
  - first–order
Balloon in driven cavity

**IIM:**

- $t = 4$
- $t = 8$
- $t = 14$

**IBM:**

- $t = 4$
- $t = 8$
- $t = 14$
Summary

- High-resolution finite volume for convective terms
- Finite difference immersed interface method for pressure Poisson problem
- Tangential component of force is spread using discrete delta functions
- Normal component of force gives jump in pressure, built into Poisson solver
- Easy modification of existing immersed boundary codes?
- Implicit method for moving the boundary
- Membrane mass can also be incorporated
Adding mass to the membrane

\[ m(s)X_{tt}(s, t) = f(s, t) - [p]\vec{n} + \mu \left[ \frac{\partial u}{\partial n} \right]. \]

Force now used in pressure and fluid solve:

\[ \tilde{f}(s, t) = f(s, t) - m(s)X_{tt}(s, t). \]

Split this into normal and tangental components.

Easy to incorporate into implicit algorithm:

**Step I3.** Replace \( \frac{1}{2}(f(X^n) + f(X^{[I]})) \) by

\[ \frac{1}{2}(f(X^n) + f(X^{[I]})) - \frac{m(s)}{\Delta t^2}(X^{[I]} - 2X^n + X^{n-1}). \]