Chapter 9
CONCLUSIONS AND FUTURE WORK

The work presented in this thesis is naturally separated into three categories: new algorithms, software, and applications. The contributions of this work in each category are discussed in the following sections. The final section of this chapter discusses future directions of the work presented in this dissertation.

9.1 A new class of wave propagation methods

The main contribution of the work presented in this dissertation is a new class of wave propagation methods to solve hyperbolic conservation laws on curved manifolds. Two novel features of these algorithms include:

1. Riemann problems are solved by decomposing modified flux differences into f-waves. These f-waves where shown to handle spatially-varying flux functions by appropriately modifying wave strengths to generate the correct solution to the generalized Riemann problem. For conservation laws on a manifold expressed in a coordinate basis, the spatially-varying flux was shown to take the form
$$\boldsymbol{f}(\boldsymbol{q}, x^k) = \sqrt{h}\,\boldsymbol{f}(\boldsymbol{q}).$$
In this case, the f-wave decomposition was shown to to naturally incorporate changing cell volumes into the strength of propagating waves.

2. Source terms of the form
$$\boldsymbol{\psi} = -\Gamma^i_{jk} T^{jk}$$
that appear for the components of vector conservation laws were incorporated into Riemann problems by using parallel transport. Each vector component of the flux is transported from cell center to cell edge prior to the solution of Riemann problems. This process sets up the modified initial data for each Riemann problem.

These two features of the algorithm incorporate all geometric influence of the manifold into the local Riemann problems. The motivation for this is that the conservation law (1.2) is a local statement of conservation. The algorithms developed in this thesis respect this and treat the conservation locally.

In addition to these features, the algorithm utilizes the solution of Riemann problems in orthonormal sets of basis vectors. The advantage of using such bases is that standard Riemann solvers based on Cartesian bases can be used directly, as long as they are modified to use f-waves. This is an advantage for systems, e.g., general relativistic magnetohydrodynamics, that have a complicated wave structure in the solution of Riemann problems. It will be useful to keep the Riemann solvers for such systems as simple as possible.

Though the main focus of the work in this thesis was on developing an algorithm for conservation laws on manifolds, this led to the development of wave propagation algorithms for general spatially-varying flux functions, and algorithms for conservation laws with general source terms. In Chapter 3, we developed a new wave propagation algorithm based on f-waves to solve conservation laws with spatially-varying flux functions. Such fluxes appear in many different physical applications. The method based on f-waves was shown to perform well on both a linear scalar equation and a nonlinear system.

In Chapter 4, we developed two different wave propagation methods to handle source terms that appear in various applications. The first method was based on quasi-steady assumptions, and was shown to perform well near a steady state. However, this method fails in some cases when the solution is dynamic. The second method, based on modified flux differences, performed well for both steady-state and dynamic solutions. The new algorithm for conservation laws on manifolds was shown to implement the second method based on modified flux differences.

9.2 Software

The algorithms developed in this thesis can be applied to many different hyperbolic conservation laws on general manifolds, making them useful to a broad class of researchers. To make use of this generality, the algorithms have been implemented in an equally general way. The software package CLAWMAN, described in Chapter 7, implements these algorithms so that the user only has to provide:

1. A Cartesian-based Riemann solver for the particular system of conservation laws needed for the application. The Riemann solver need only be modified to decompose the flux difference into f-waves as described in Chapter 3.

2. The routine `hfunct.f90`, shown in Table 7.3, which sets the components of the metric tensor $[h_{jk}]$. The metric describes the geometry on $\mathscr{M}$.

3. The routines `user_module.f90` and `init_user_params.f90` described in Section 7.2.2 that set any user-defined data structures and parameters.

4. The routines that set initial and boundary conditions.

This code is freely available and can be downloaded at

`http://www.amath.washington.edu/~claw/clawman`

with applications included. The number of available applications will continue to grow as the code is applied to more conservation laws and different manifolds.

9.3 Applications

We have discussed many different applications throughout this thesis to verify that the proposed methods work well on practical problems. In Chapter 3, we presented results generated by the method based on f-waves for two applications. The first was a modified Burgers' equation with a flux that depended linearly on the spatial variable. In the second application, we presented results for the quasi-one-dimensional flow of a gas in a duct with variable cross section. In both examples the method based on f-waves was shown to perform well.

In Chapter 4, we presented results generated by two wave propagation methods for conservation laws with source terms. The examples in this chapter involved the Euler equations of a gas under the influence of a stationary gravitational field. An isothermal equilibrium was set up and pressure perturbations were set in motion on top of that equilibrium. The solutions of the two new methods were compared with a solution generated by a split method. Both of these new methods were shown to perform better than the split method for this example.

The examples that test the main algorithm of this thesis were presented in Chapter 6. The purpose of these examples was to show that the new algorithm for conservation laws on manifolds performs well on different systems of equations on various manifolds. The examples shown in Chapter 6 where done using CLAWMAN. In the first example, the Euler equations were solved in a curvilinear set of coordinates that cover $\mathbb{R}^2$. This example shows that the method can be used for conservation laws in curvilinear coordinates. The next set of examples in Chapter 6 involve a linear system of equations on a Riemannian manifold. These examples demonstrate that the new algorithm performs well on curved manifolds.

The motivation for developing wave propagation algorithms for conservation laws on curved manifolds has been the solution of the equations governing relativistic test fluids. The geometry of the spacetime manifold affects the flow of such fluids. Chapter 8 includes several applications of the methods discussed in this thesis to standard problems for relativistic fluids. First, two special relativistic Riemann problems are presented. The first Riemann problem has a strong transonic rarefaction wave, and the second involves a thin dense shell created by a strong shock wave. The method performs well on both problems.

The second example in Chapter 8 presents the spherical accretion of dust into a Schwarzschild black hole. In this case, the manifold is curved by the mass of the black hole. Finally, a multidimensional application is presented for an isentropic, irrotational perfect fluid flowing past a relativistic hard sphere. In this example, the speed of sound is equal to the speed of light.

9.4 Future directions of this work

The algorithms developed in this dissertation, and the implementation of them in CLAWMAN is general so that they can be applied to different systems of conservation laws on a variety of manifolds. This generality allows for many directions of future research.

Most of the work presented in this thesis concerned developing general algorithms for solving conservation laws on Riemannian manifolds. There are many classical systems that can be investigated with the methods developed in the previous chapters. For example, the propagation of elastic waves on the curved surface of a membrane or shell would be an interesting application of the method.

The main motivation for developing these algorithms was to solve the equations of general relativistic hydrodynamics. Since there are only a few high-resolution finite volume methods based on the solution of Riemann problems that have been applied to relativistic hydrodynamics, there are many directions for further application of the new methods in this field. The applications presented in Chapter 8 are only a preliminary set of relativistic systems to which the new methods can be applied.

Numerical results for the accretion of relativistic gases onto moving black holes have been reported in several papers, e.g., see [6] and [23]. The methods presented in this thesis will be applied to such problems and the results will be compared with those found in the literature.

In the relativistic accretion problems, the curvature of the manifold is centered on the black hole. The boundaries of the computation are easy to implement if they are out in the flat regions away from the hole. Unfortunately, the grid must be fairly large to accomplish this while maintaining good grid refinement. Eventually, the methods presented in this thesis will be combined with the adaptive refinement capabilities of the software package BEARCLAW. This will enable the placement of boundaries in flat regions of black hole spacetimes. It will also allow the investigation of the breakup of small objects like single stars as they plunge into the central regions of black holes.

Finally, the implementation in CLAWMAN is based on a metric that is independent of time. However, there is no reason that the metric cannot be dependent on time. This would allow the methods developed in this thesis to be applied to moving surfaces, or the full Einstein equations in which the metric tensor is evolving in time with the fluid variables. Bardeen and Buchman [7] are applying wave propagation algorithms to the vacuum Einstein equations. Eventually, the goal is to combine the methods for relativistic test fluids developed in this thesis with wave propagation methods of the form they are using to produce an algorithm that is capable of solving the full Einstein equations.