# Finite Volume Methods for Hyperbolic Problems

## Acoustics in Heterogeneous Media

- One space dimension
- Reflection and transmission at interfaces
- Non-conservative form, Riemann problems
- Two space dimensions
- Transverse Riemann solver
- Some examples

# One-dimensional Elasticity

Compressional waves similar to acoustic waves in gas.

Notation:

$$X(x,t) = \text{ location of particle indexed by } x \text{ in the reference (undeformed) configuration}$$

$$X(x,0) = x \text{ if initially undeformed}$$

$$\epsilon(x,t) = X_x(x,t) - 1 = \text{ strain}$$

$$u(x,t) = \text{ velocity of particle indexed by } x$$

$$\sigma(\epsilon) = \text{ stress–strain relation}$$

$$\rho = \text{ density}$$

# Linear elasticity

Hyperbolic conservation law:

$$\epsilon_t - u_x = 0 \qquad \text{since } \epsilon_t = X_{xt} = X_{tx} = u_x$$
$$\rho u_t - \sigma_x = 0 \qquad \text{conservation of momentum, } F = ma$$

Linear stress-strain relation (Hooke's law):

$$\sigma(\epsilon) = K\epsilon$$

where $K$ is the bulk modulus of compressibility.

Then

$$\begin{aligned} \sigma_t - K u_x &= 0 \\ u_t - (1/\rho)\sigma_x &= 0 \end{aligned} \qquad A = \begin{bmatrix} 0 & -K \\ -1/\rho & 0 \end{bmatrix}$$

Eigenvalues: $\lambda = \pm\sqrt{K/\rho}$ as in acoustics.

(Equivalent to acoustics with $\sigma = -p$)

# Elasticity in heterogeneous material

Suppose $\rho(x),\ \sigma(\epsilon, x)$ vary with $x$

Conservative form:

$$
\begin{aligned}
&\epsilon_t - u_x = 0 \\
&(\rho(x)u)_t - \sigma(\epsilon, x)_x = 0
\end{aligned}
$$

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Linear stress-strain relation (Hooke's law):

$$\sigma(\epsilon, x) = K(x)\epsilon$$

Non-conservative variable-coefficient linear system:

$$\begin{aligned} &\sigma_t - K(x)u_x = 0 \\ &u_t - (1/\rho(x))\sigma_x = 0 \end{aligned} \qquad A = \begin{bmatrix} 0 & -K(x) \\ -1/\rho(x) & 0 \end{bmatrix}$$

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Variable coefficient acoustics: $p = -\sigma$

# Wave propagation in heterogeneous medium

Multiply system

$$q_t + A(x)q_x = 0$$

by $R^{-1}(x)$ on left to obtain

$$R^{-1}(x)q_t + R^{-1}(x)A(x)R(x)\,R^{-1}(x)q_x = 0$$

or

$$(R^{-1}(x)q)_t + \Lambda(x)\,\left[(R^{-1}(x)q)_x - R_x^{-1}(x)q\right] = 0$$

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Let $w(x,t) = R^{-1}(x)q(x,t)$ (characteristic variable).

There is a coupling term on the right: Note typo in (9.51)

$$w_t + \Lambda(x)\,w_x = \Lambda(x)R_x^{-1}(x)R(x)w$$

If the eigenvectors vary with $x$ (i.e. if $R_x \neq 0$)
then waves in other families are generated (e.g. reflections)

# Wave propagation in heterogeneous medium

Linear system $q_t + A(x)q_x = 0$. For acoustics:

$$A = \begin{bmatrix} 0 & K(x) \\ 1/\rho(x) & 0 \end{bmatrix} \qquad q = \begin{bmatrix} p \\ u \end{bmatrix}.$$

eigenvalues: $\lambda^1 = -c(x), \qquad \lambda^2 = +c(x),$

where $c(x) = \sqrt{K(x)/\rho(x)} =$ local speed of sound.

eigenvectors: $r^1(x) = \begin{bmatrix} -Z(x) \\ 1 \end{bmatrix}, \qquad r^2(x) = \begin{bmatrix} Z(x) \\ 1 \end{bmatrix}$

where $Z(x) = \rho c = \sqrt{\rho K} =$ impedance.

# Transmission and reflection coefficients

Consider an interface between two materials with constant properties in each.

$$\rho_\ell,\ K_\ell \implies c_\ell = \sqrt{\rho_\ell/K_\ell},\ Z_\ell = \sqrt{\rho_\ell K_\ell}$$

$$\rho_r\ K_r \implies c_r = \sqrt{\rho_r/K_r},\ Z_r = \sqrt{\rho_r K_r}$$

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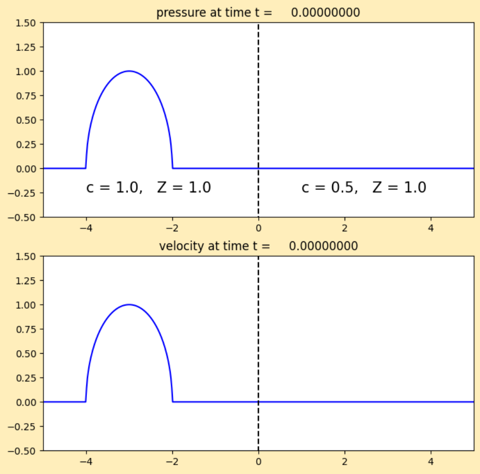
More generally, wave is partly transmitted and partly reflected,

$$C_T = \frac{2Z_r}{Z_\ell + Z_r}, \qquad C_R = \frac{Z_r - Z_\ell}{Z_\ell + Z_r}.$$

# Right-going simple wave with $Z_\ell = Z_r$




Note $p$ and $u$ are not conserved, but they are always continuous.

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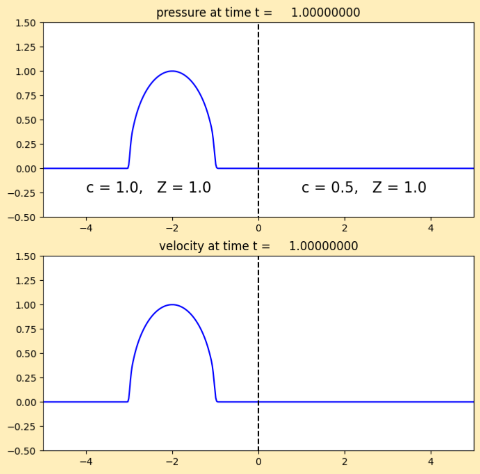


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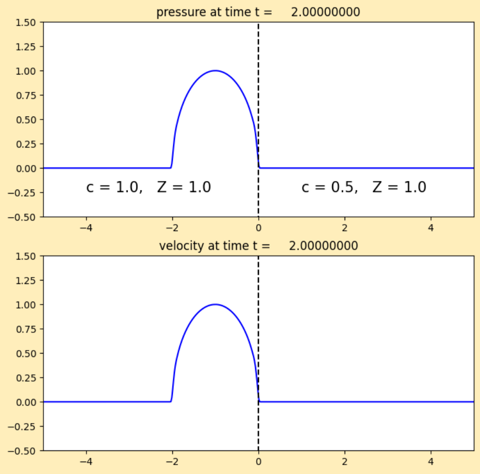


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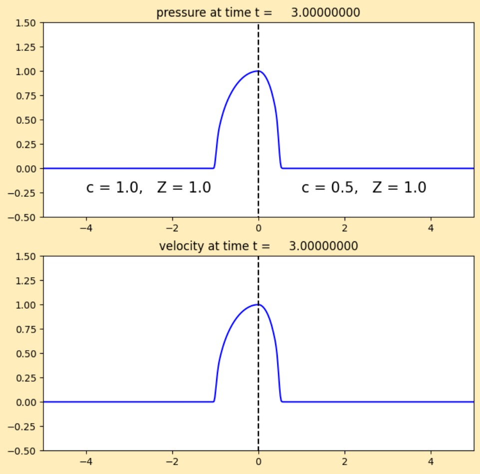


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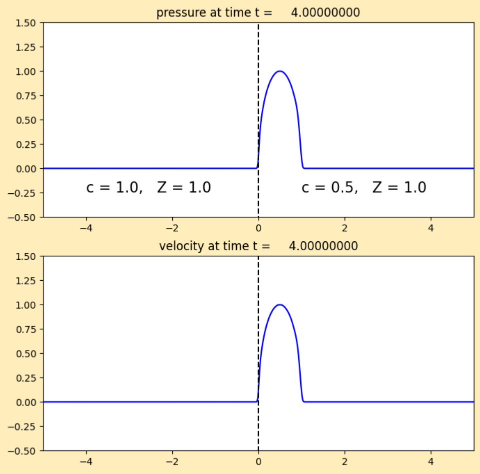


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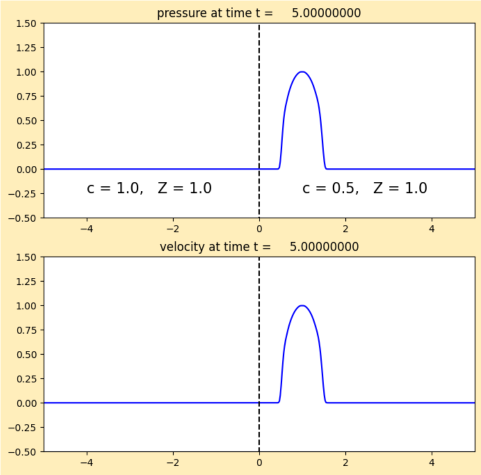


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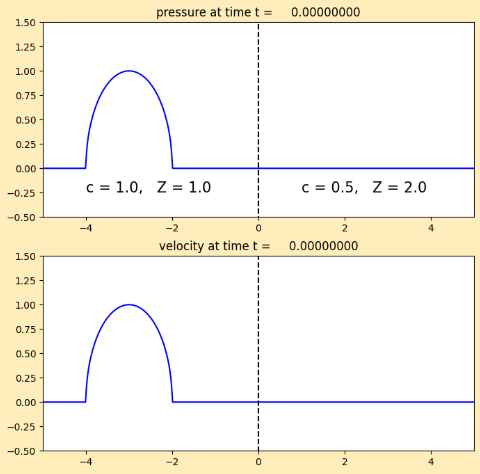
# Transmitted/reflected wave with $Z_\ell \neq Z_r$




$$C_T = \frac{2Z_r}{Z_\ell + Z_r} = \frac{4}{3}$$

$$C_R = \frac{Z_r - Z_\ell}{Z_\ell + Z_r} = \frac{1}{3}$$

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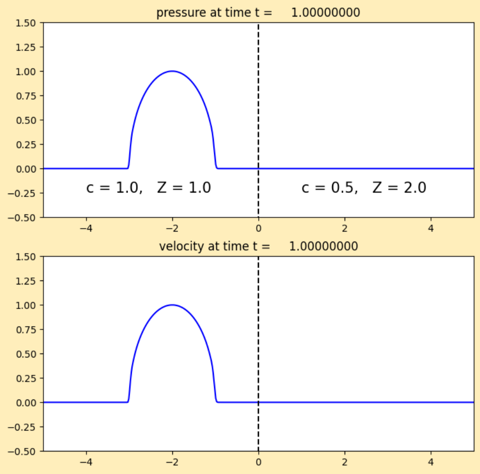


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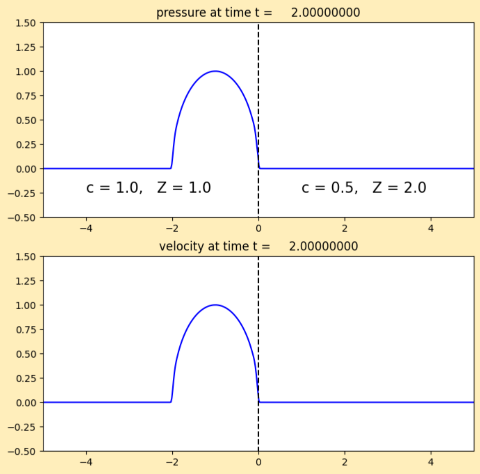


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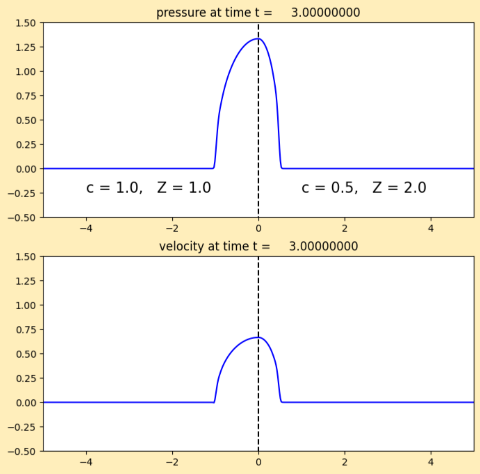


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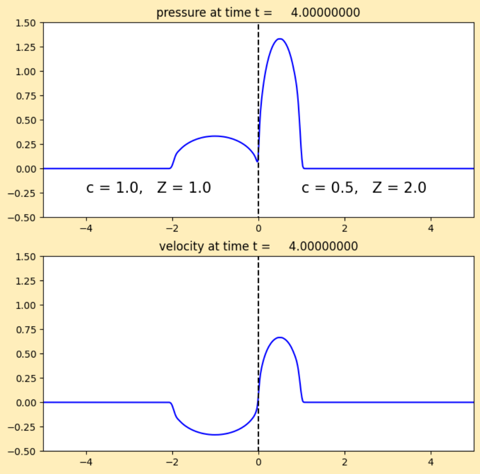


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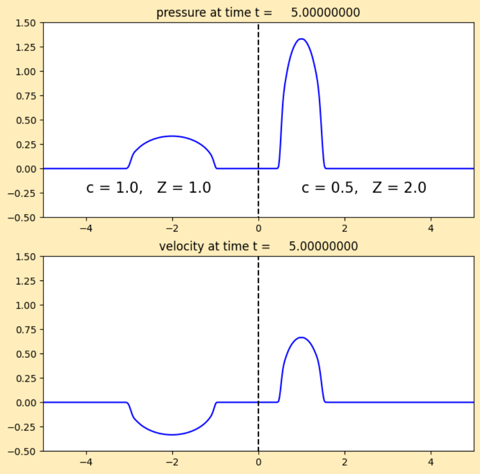


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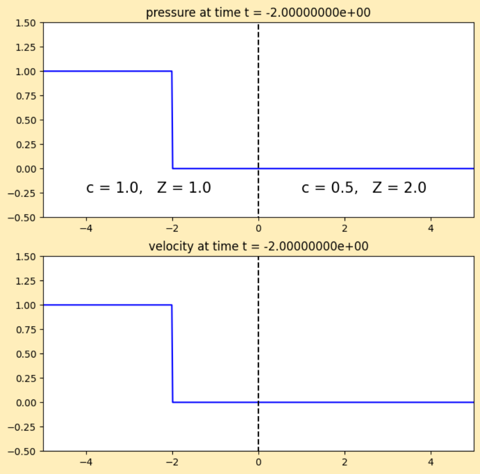


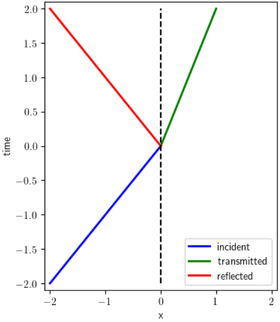


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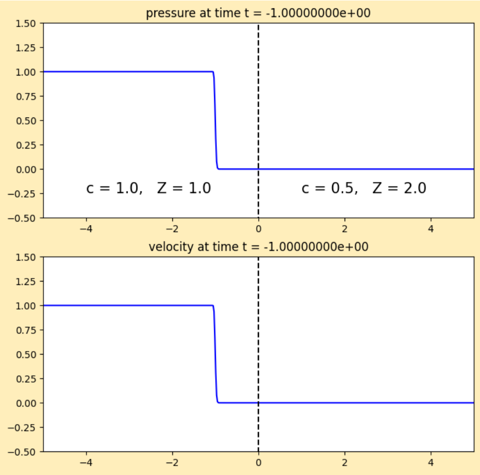


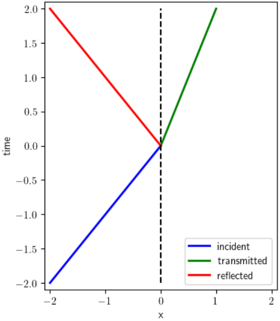


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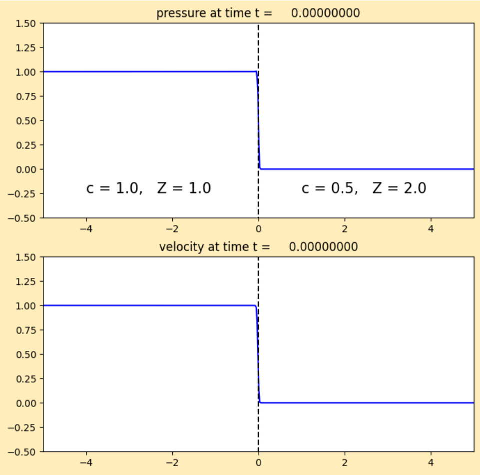


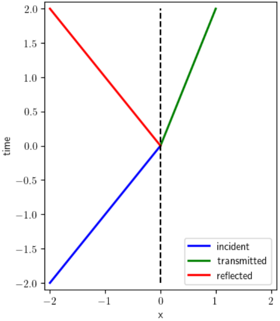


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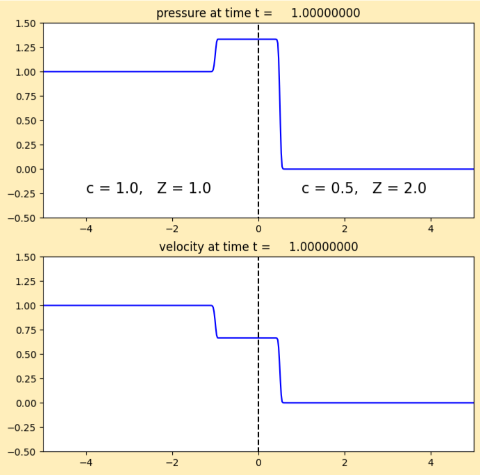


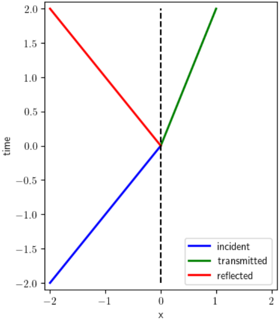


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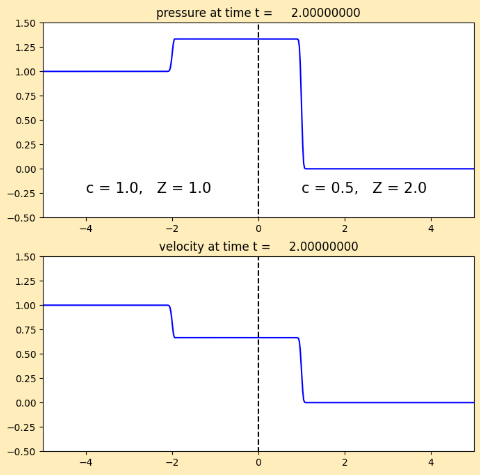


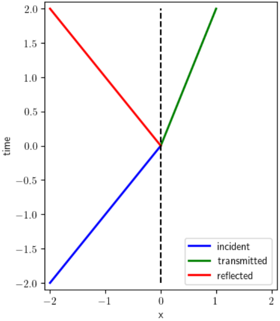


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# Riemann problem for heterogeneous medium

Jump discontinuity in $q(x, 0)$ and in $K(x)$ and $\rho(x)$.

Decompose jump in $q$ as linear combination of eigenvectors:

- left-going waves: eigenvectors for material on left,
- right-going waves: eigenvectors for material on right.

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$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \qquad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

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Riemann solution: decompose

$$q_r - q_l = \alpha^1 \begin{bmatrix} -Z_l \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix} = \mathcal{W}^1 + \mathcal{W}^2$$

The waves propagate with speeds $s^1 = -c_l$ and $s^2 = c_r$.

# Wave propagation in heterogeneous medium

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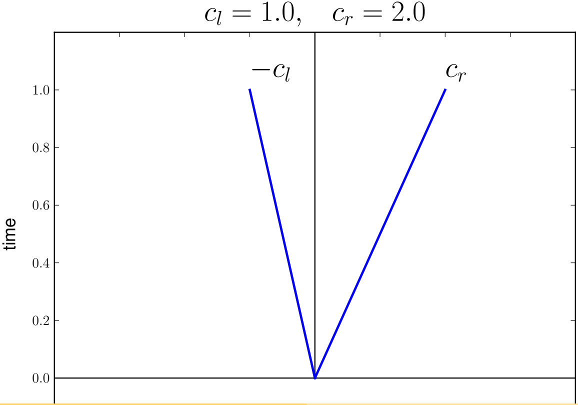

# Riemann problem for interface

$$q_r - q_\ell = \alpha^1 \begin{bmatrix} -Z_\ell \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix}.$$

gives the linear system

$$R_{\ell r}\,\alpha = q_r - q_\ell,$$

where

$$R_{\ell r} = \begin{bmatrix} -Z_\ell & Z_r \\ 1 & 1 \end{bmatrix} \qquad \Longrightarrow \; R_{\ell r}^{-1} = \frac{1}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix}$$

So

$$\begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix} = \frac{1}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix} \begin{bmatrix} p_r - p_\ell \\ u_r - u_\ell \end{bmatrix}.$$

# 2-wave hitting interface as a Riemann problem

Incident wave:

$$q_r - q_\ell = \beta r_\ell^2 = \beta \begin{bmatrix} Z_\ell \\ 1 \end{bmatrix},$$

then Riemann solution gives

$$\begin{aligned} \alpha &= R_{lr}^{-1}(q_r - q_\ell) \\ &= \frac{\beta}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix} \begin{bmatrix} Z_\ell \\ 1 \end{bmatrix} \\ &= \frac{\beta}{Z_\ell + Z_r} \begin{bmatrix} Z_r - Z_\ell \\ 2Z_\ell \end{bmatrix}. \end{aligned}$$

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$$\implies \alpha^1 = \left(\frac{Z_r - Z_\ell}{Z_\ell + Z_r}\right)\beta \quad \text{and} \quad \alpha^2 = \left(\frac{2Z_\ell}{Z_\ell + Z_r}\right)\beta$$

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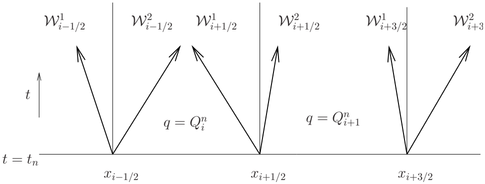
Pressure jump in reflected wave: $c_R \beta Z_\ell$

Pressure jump in transmitted wave: $c_T \beta Z_\ell$

# Godunov's method — variable coefficient acoustics




$$
\begin{aligned}
Q_i - Q_{i-1} &= \begin{bmatrix} p_i - p_{i-1} \\ u_i - u_{i-1} \end{bmatrix} \\
&= \alpha^1_{i-1/2} \begin{bmatrix} -\rho_{i-1} c_{i-1} \\ 1 \end{bmatrix} + \alpha^2_{i-1/2} \begin{bmatrix} \rho_i c_i \\ 1 \end{bmatrix} \\
&= \alpha^1_{i-1/2} r^1_{i-1} + \alpha^2_{i-1/2} r^2_i \\
&= \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2}
\end{aligned}
$$

# 2D Acoustics in Heterogeneous Media

$$q_t + A(x,y)q_x + B(x,y)q_y = 0,$$

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & K(x,y) & 0 \\ 1/\rho(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ 1/\rho(x,y) & 0 & 0 \end{bmatrix}.$$

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Riemann problem in $x$:

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \qquad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix},$$

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$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \qquad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \alpha^1 &= (-\Delta Q^1 + Z_{ij}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\ \alpha^2 &= \Delta Q^3, \\ \alpha^3 &= (\Delta Q^1 + Z_{i-1,j}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}). \end{aligned}$$

Wave speeds: $s^1 = -c_{i-1,j}, \quad s^2 = 0, \quad s^3 = c_{ij}$

Only need to propagate and apply limiters to $\mathcal{W}^1,\ \mathcal{W}^3$.

# Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in $x$ or $y$ direction.
In latter case splitting is done using $B$ instead of $A$.
This is all that's required for dimensional splitting.

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In latter case splitting is done using $B$ instead of $A$.
This is all that's required for dimensional splitting.

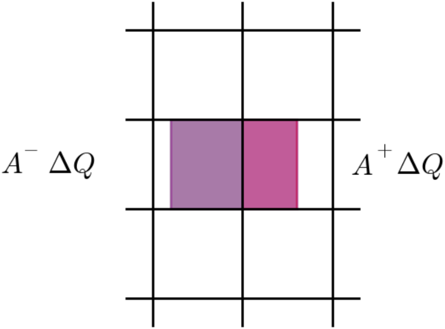
Transverse Riemann solver `rpt2.f`
Decomposes $\mathcal{A}^+\Delta Q$ into $\mathcal{B}^-\mathcal{A}^+\Delta Q$ and $\mathcal{B}^+\mathcal{A}^+\Delta Q$ by splitting this vector into eigenvectors of $B$.

(Or splits vector into eigenvectors of $A$ if `ixy=2`.)

# Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



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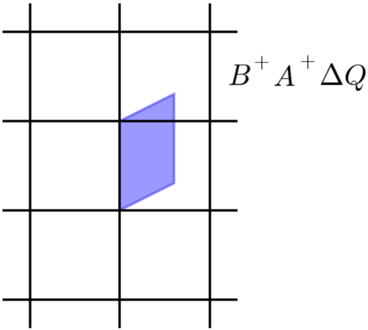

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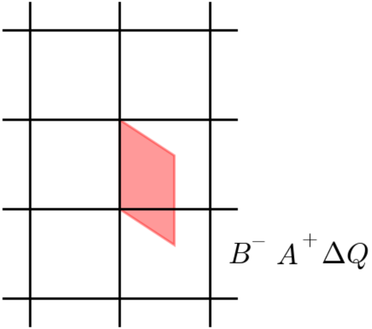

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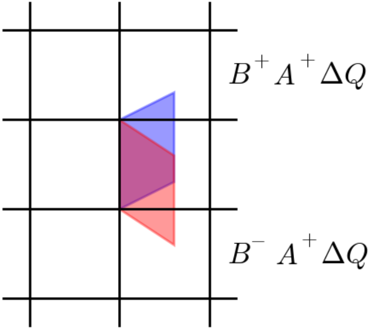
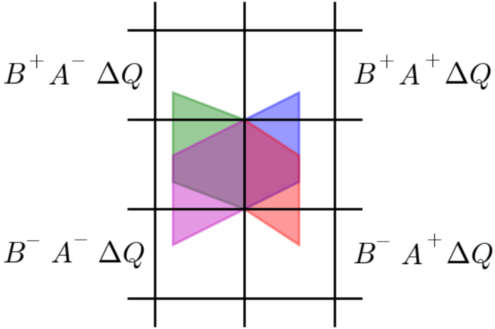


# Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



# Transverse solver for 2D Acoustics

Solving Riemann problem in $x$ gives waves and fluctuations
$\mathcal{A}^-\Delta Q_{i-1/2,j}$, $\mathcal{A}^+\Delta Q_{i-1/2,j}$.

For $\mathcal{B}^-\mathcal{A}^+\Delta Q_{i-1/2,j}$ we want downward-going part of $\mathcal{A}^+\Delta Q_{i-1/2,j}$,
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$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{i,j-1} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{ij} \\ 0 \\ 1 \end{bmatrix},$$

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with speeds $-c_{i,j-1},\ 0,\ c_{ij}$ respectively.

Only use downward-going part:

$$\beta^1 = \left(-(\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{ij}\right) / (Z_{i,j-1} + Z_{ij}),$$

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$$\mathcal{B}^-\mathcal{A}^+\Delta Q_{i-1/2,j} = -c_{i,j-1}\beta^1 \begin{bmatrix} -Z_{i,j-1} \\ 0 \\ 1 \end{bmatrix}$$

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$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{ij} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix},$$

with speeds $-c_{ij},\ 0,\ c_{i,j+1}$ respectively.

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with speeds $-c_{ij},\ 0,\ c_{i,j+1}$ respectively.

Only use upward-going part:

$$\beta^3 = \left((\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{i,j+1}\right) / (Z_{ij} + Z_{i,j+1})$$

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$$\mathcal{B}^+\mathcal{A}^+\Delta Q_{i-1/2,j} = c_{i,j+1}\beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix}$$

# Cell averaging material parameters

To solve a variable coefficient problem on a grid,
need to average material parameters onto grid cell.

For acoustics with $\rho(x, y),\ K(x, y)$, on Cartesian grid:

Can use mean value of density:

$$\rho_{ij} = \frac{1}{\Delta x \Delta y} \iint \rho(x, y)\, dx, dy$$

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But need to use harmonic average of bulk modulus:

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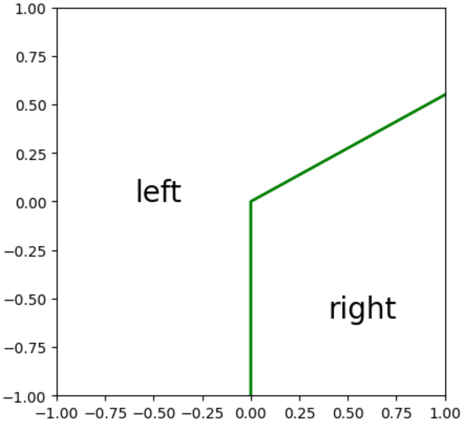
Then $c_{ij} = \sqrt{K_{ij}/\rho_{ij}}, \quad Z_{ij} = \sqrt{K_{ij}\rho_{ij}}$

# Acoustic wave hitting an interface in 2D

Example from Figure 21.1:

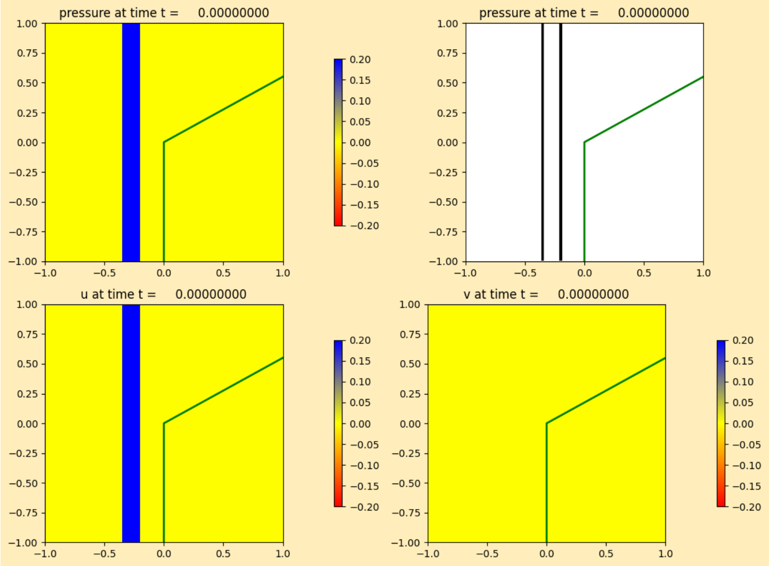



$$\begin{aligned} \rho_\ell &= 1 & \rho_r &= 1 \\ K_\ell &= 1 & K_r &= 0.25 \\ c_\ell &= 1 & c_r &= 0.5 \\ Z_\ell &= 1 & Z_r &= 0.5 \end{aligned}$$

$$\begin{aligned} C_T &= \frac{2Z_r}{Z_\ell + Z_r} \\ &= 2/3 \end{aligned}$$

$$\begin{aligned} C_R &= \frac{Z_r - Z_\ell}{Z_\ell + Z_r} \\ &= -1/3 \end{aligned}$$

# Acoustic wave hitting an interface in 2D



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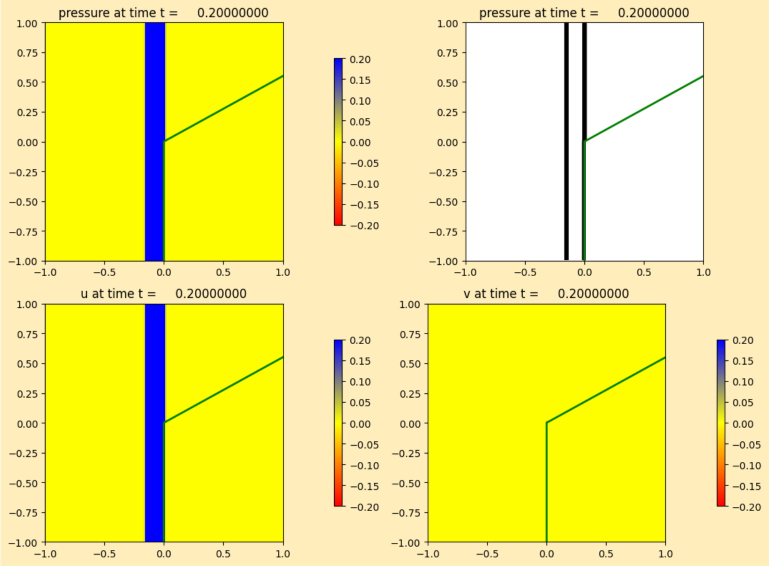

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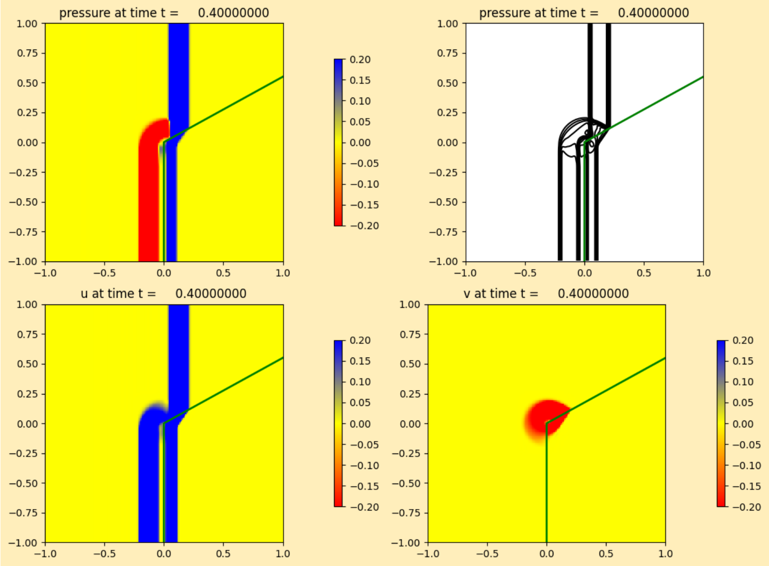

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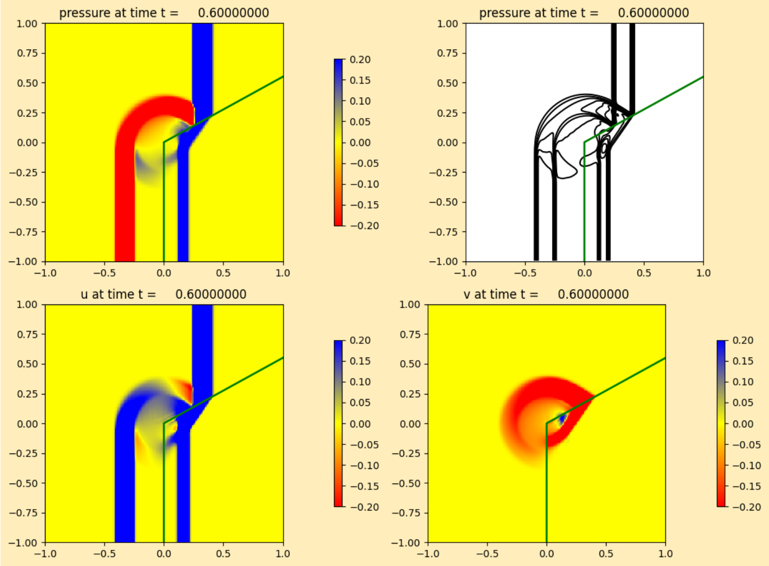

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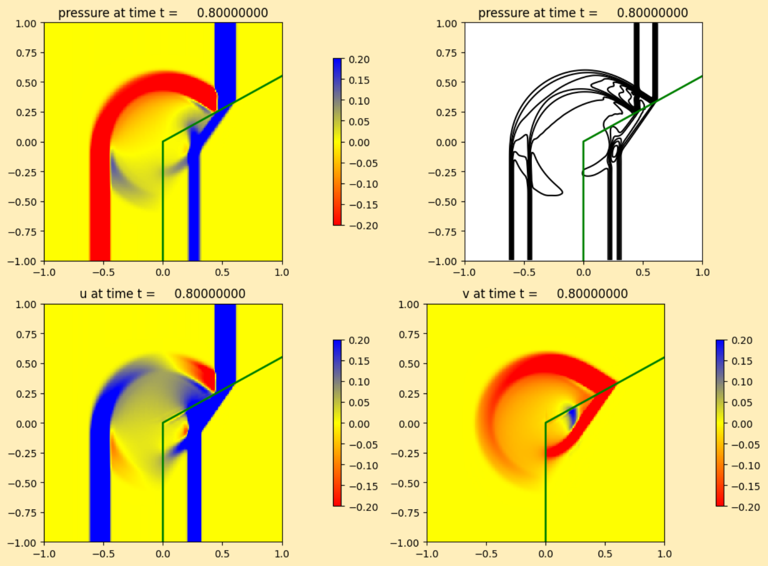
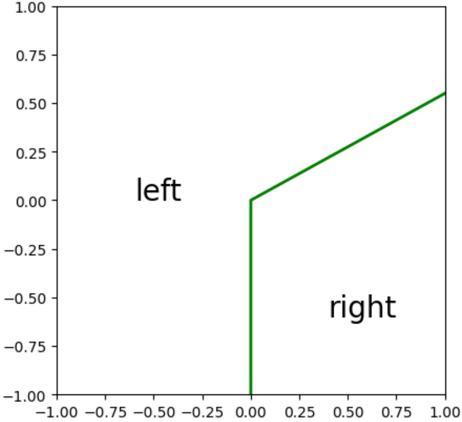


With nearly-incompressible material on right (≈ solid wall)



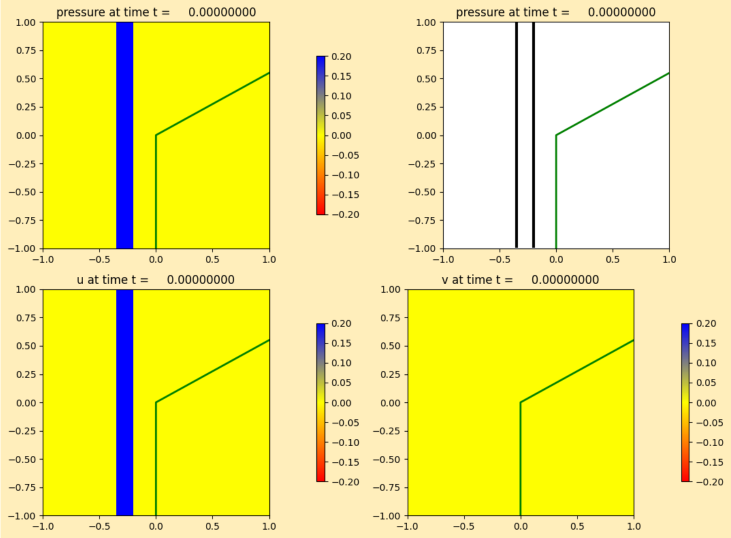

$$\begin{aligned} \rho_\ell &= 1 & \rho_r &= 10^4 \\ K_\ell &= 1 & K_r &= 10^{-8} \\ c_\ell &= 1 & c_r &= 10^{-6} \\ Z_\ell &= 1 & Z_r &= 0.01 \end{aligned}$$

$$\begin{aligned} C_T &= \tfrac{2Z_r}{Z_\ell+Z_r} \\ &\approx 0.02 \end{aligned}$$

$$\begin{aligned} C_R &= \tfrac{Z_r-Z_\ell}{Z_\ell+Z_r} \\ &\approx -0.98 \end{aligned}$$

# Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)



# Acoustic wave hitting an interface in 2D

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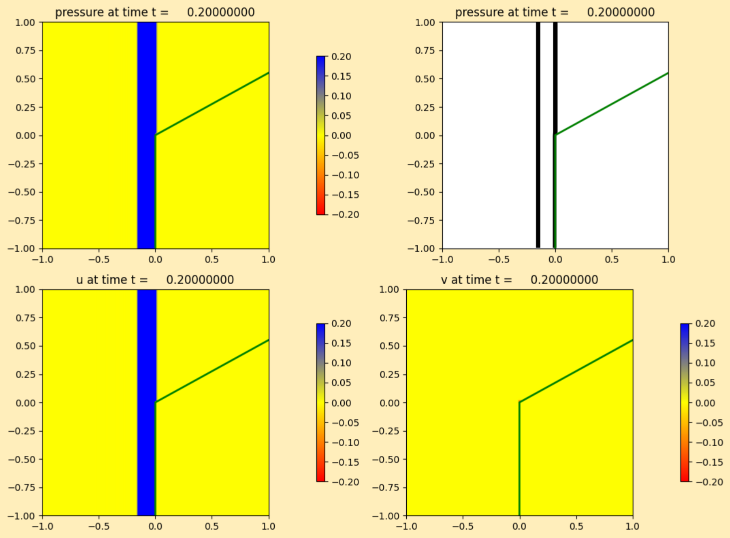

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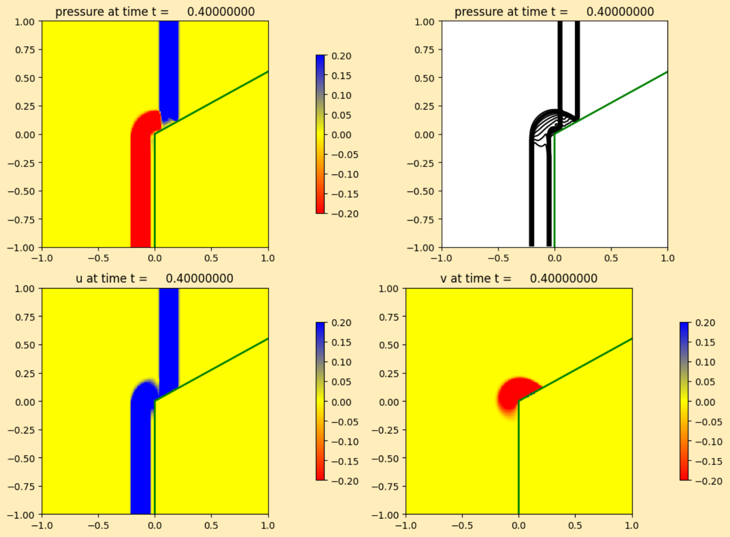

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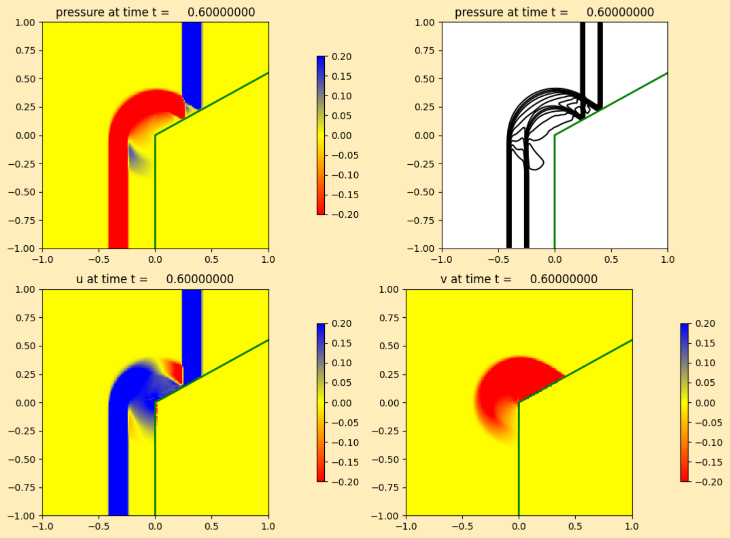

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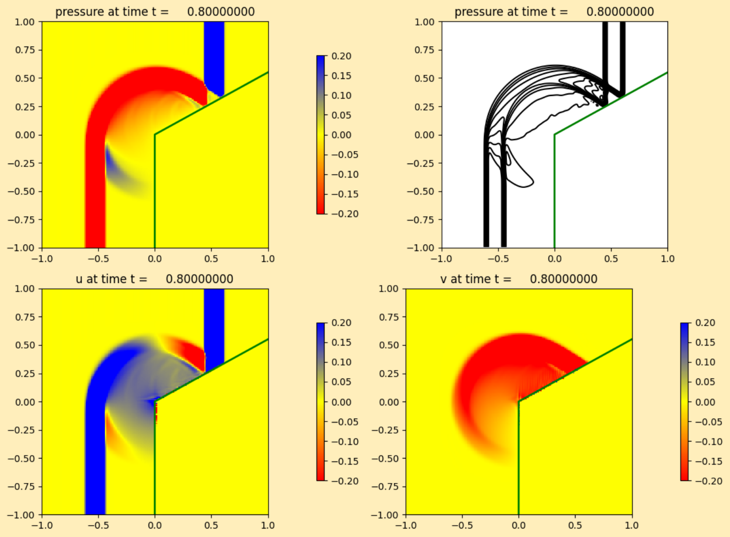
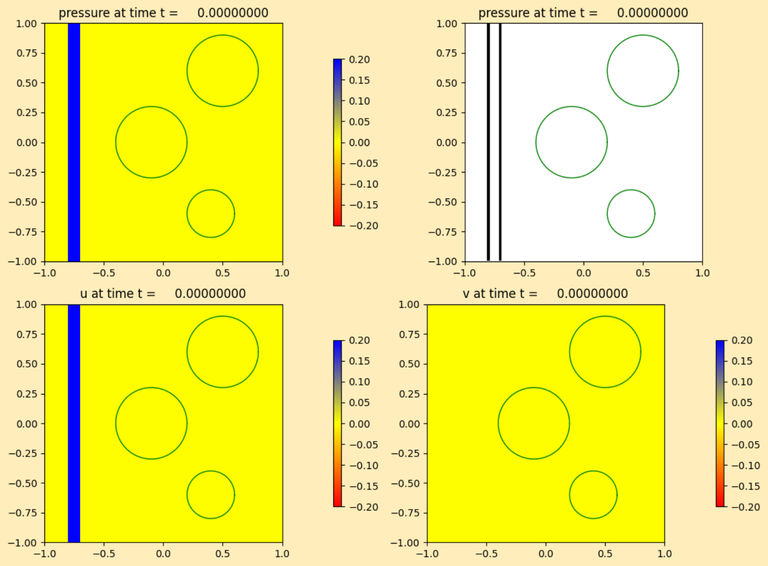


# Acoustic wave hitting circular inclusions



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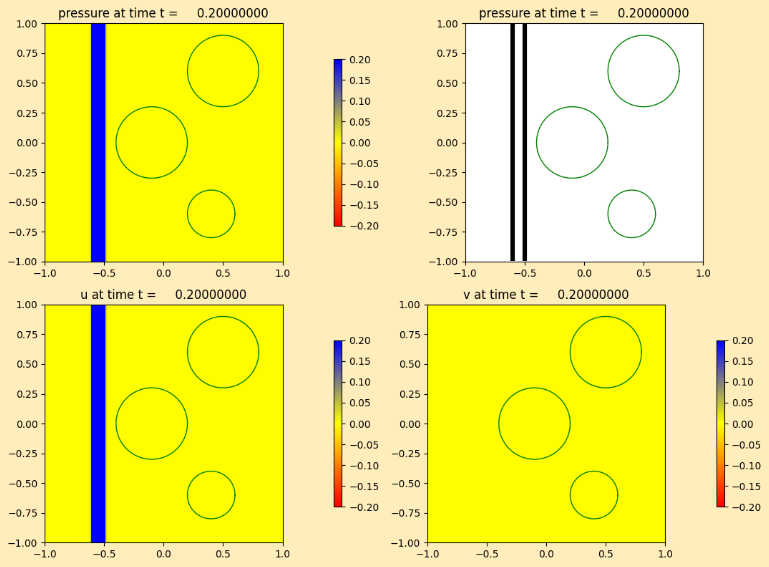

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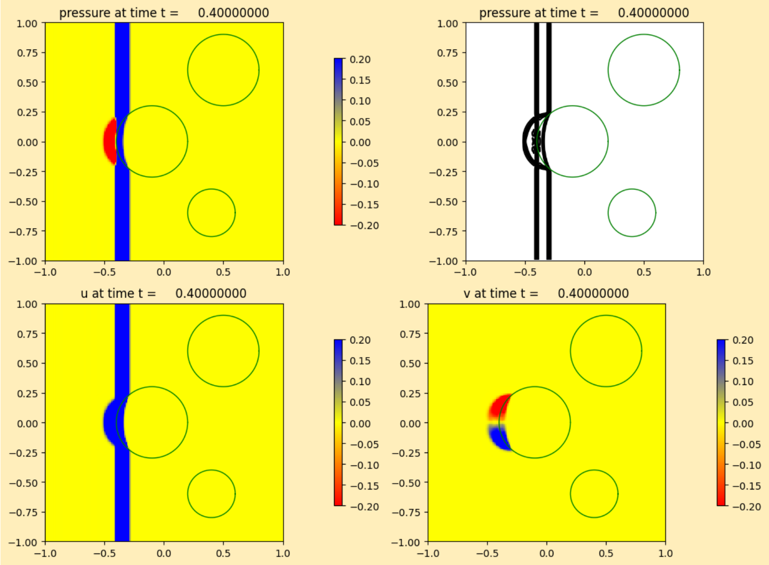

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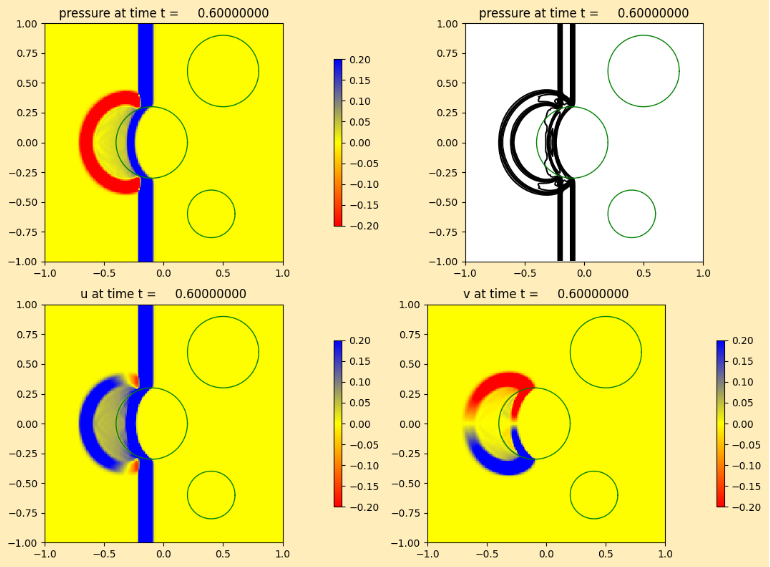

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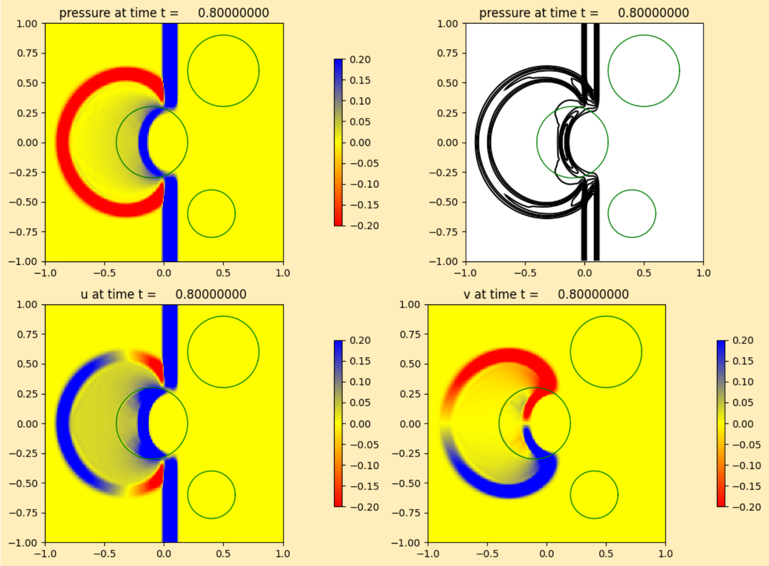

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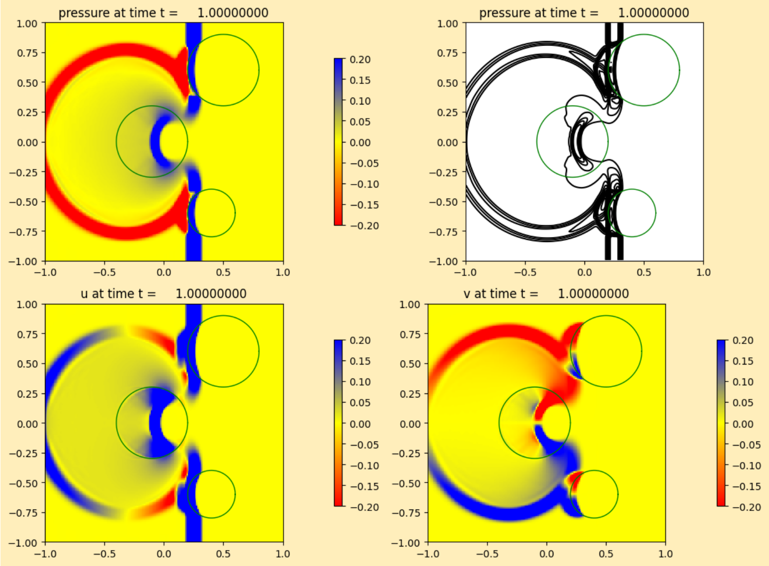

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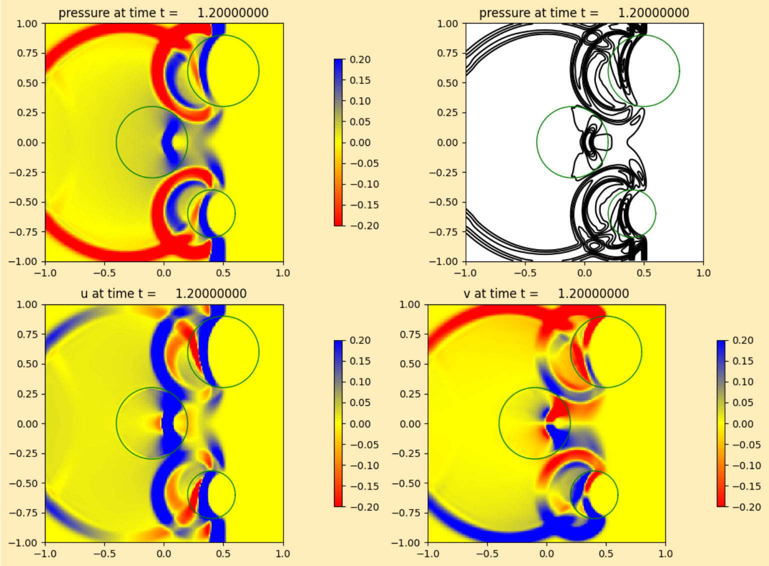

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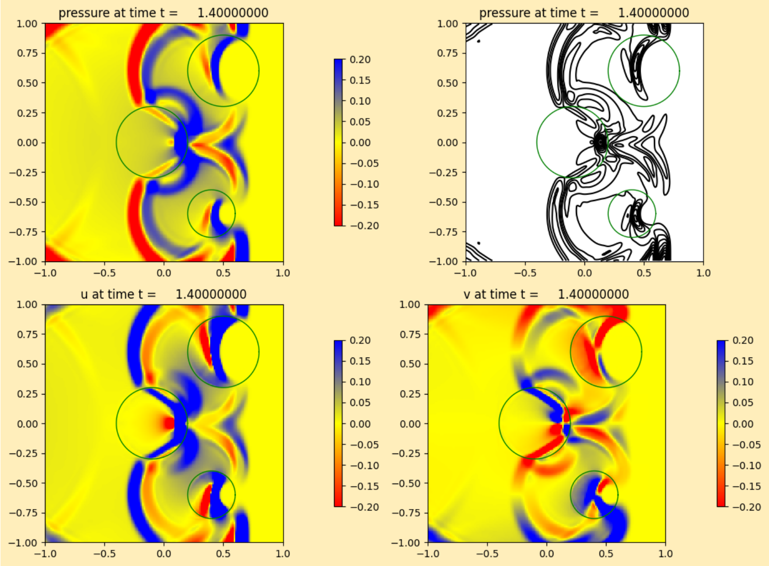

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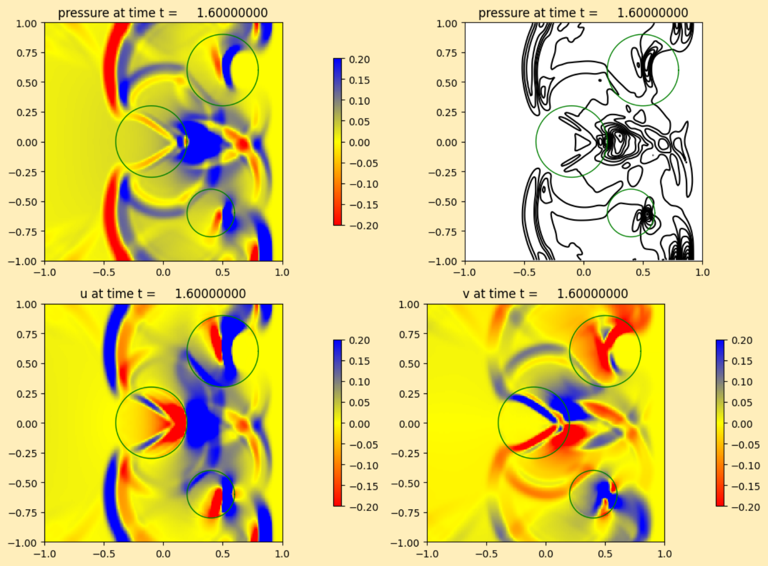

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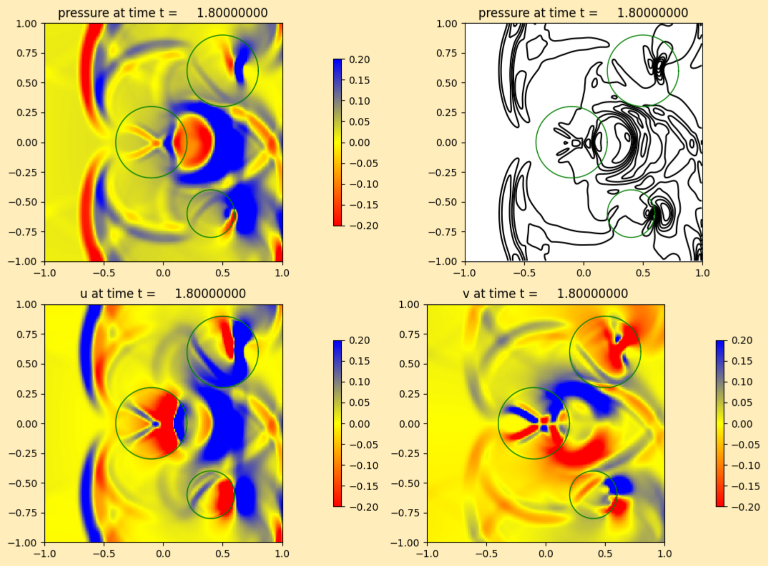

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