Finite Volume Methods for Hyperbolic Problems

Fractional Step Methods

- Dimensional splitting (Chapter 19)
- Fractional steps for source terms (Chapter 17)
- Godunov and Strang splitting
- Cross-derivatives in 2D hyperbolic problems
- Upwind splitting of ABq_{yx} and BAq_{xy}

Fractional steps for source terms

Conservation law with source term (balance law):

$$q_t(x,t) + f(q(x,t))_x = \psi(q(x,t))$$

ψ could depend on (x,t) explicitly too.

Fractional steps for source terms

Conservation law with source term (balance law):

$$q_t(x,t) + f(q(x,t))_x = \psi(q(x,t))$$

ψ could depend on (x,t) explicitly too.

Fractional step (time splitting) method:

To advance full solution by Δt, alternate between:

- $q_t(x,t) + f(q(x,t))_x = 0$ with high-resolution method,
- $q_t(x,t) = \psi(q(x,t)$, an ODE in each grid cell

Fractional steps for source terms

Conservation law with source term (balance law):

$$q_t(x,t) + f(q(x,t))_x = \psi(q(x,t))$$

ψ could depend on (x,t) explicitly too.

Fractional step (time splitting) method:

To advance full solution by Δt, alternate between:

- $q_t(x,t) + f(q(x,t))_x = 0$ with high-resolution method,
- $q_t(x,t) = \psi(q(x,t)$, an ODE in each grid cell

Source term in Clawpack: Provide `src1.f90` in 1d
or `src2.f90` in 2d that advances Q in each cell by time Δt.

Set `clawdata.src_split = 1` (or `= 2` for Strang splitting)

Dimensional Splitting

Hyperbolic system in 2d: $q_t + Aq_x + Bq_y = 0$

Use Cartesian grid and alternate between:

x-sweeps : $q_t + Aq_x = 0$

y-sweeps : $q_t + Bq_y = 0.$

Use one-dimensional high-resolution methods for each.

Dimensional Splitting

Hyperbolic system in 2d: $q_t + Aq_x + Bq_y = 0$

Use Cartesian grid and alternate between:

x-sweeps : $q_t + Aq_x = 0$

y-sweeps : $q_t + Bq_y = 0.$

Use one-dimensional high-resolution methods for each.

- Easy to extend good 1D methods to 2D or 3D.
- Often very effective and efficient.
- May suffer from lack of isotropy.
- May be hard to use with AMR, complex geometry.

Dimensional Splitting

Hyperbolic system in 2d: $q_t + Aq_x + Bq_y = 0$

Use Cartesian grid and alternate between:

x-sweeps : $q_t + Aq_x = 0$

y-sweeps : $q_t + Bq_y = 0.$

Use one-dimensional high-resolution methods for each.

- Easy to extend good 1D methods to 2D or 3D.
- Often very effective and efficient.
- May suffer from lack of isotropy.
- May be hard to use with AMR, complex geometry.

Alternative: Unsplit methods.

Fractional step method for a linear PDE

$$q_t = (\mathcal{A} + \mathcal{B})q \qquad \text{dimensional splitting: } \mathcal{A} = A\partial_x, \;\; \mathcal{B} = B\partial_y.$$

Then

$$q_{tt} = (\mathcal{A} + \mathcal{B})q_t = (\mathcal{A} + \mathcal{B})^2 q,$$

and so

$$\begin{aligned} q(x, \Delta t) &= q(x,0) + \Delta t(\mathcal{A} + \mathcal{B})q(x,0) + \frac{1}{2}\Delta t^2(\mathcal{A} + \mathcal{B})^2 q(x,0) + \cdots \\ &= \left(I + \Delta t(\mathcal{A} + \mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A} + \mathcal{B})^2 + \cdots\right) q(x,0) \end{aligned}$$

Solution operator: $q(x, \Delta t) = e^{\Delta t(\mathcal{A} + \mathcal{B})} q(x,0)$.

Fractional step method for a linear PDE

$$q_t = (\mathcal{A} + \mathcal{B})q \qquad \text{dimensional splitting: } \mathcal{A} = A\partial_x, \;\; \mathcal{B} = B\partial_y.$$

Then

$$q_{tt} = (\mathcal{A} + \mathcal{B})q_t = (\mathcal{A} + \mathcal{B})^2 q,$$

and so

$$\begin{aligned} q(x, \Delta t) &= q(x,0) + \Delta t(\mathcal{A} + \mathcal{B})q(x,0) + \frac{1}{2}\Delta t^2(\mathcal{A} + \mathcal{B})^2 q(x,0) + \cdots \\ &= \left(I + \Delta t(\mathcal{A} + \mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A} + \mathcal{B})^2 + \cdots\right) q(x,0) \end{aligned}$$

Solution operator: $q(x, \Delta t) = e^{\Delta t(\mathcal{A}+\mathcal{B})} q(x,0)$.

With the fractional step method, we instead compute

$$q^*(x, \Delta t) = e^{\Delta t \mathcal{A}} q(x,0),$$

and then

$$q^{**}(x, \Delta t) = e^{\Delta t \mathcal{B}} q^*(x, \Delta t) \;\; = e^{\Delta t \mathcal{B}} e^{\Delta t \mathcal{A}} q(x,0).$$

Splitting error

$$q(x,\Delta t) - q^{**}(x,\Delta t) = \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right) q(x,0)$$

Combining 2 steps gives:

$$\begin{aligned} q^{**}(x,\Delta t) &= \left(I + \Delta t\mathcal{B} + \frac{1}{2}\Delta t^2\mathcal{B}^2 + \cdots\right)\left(I + \Delta t\mathcal{A} + \frac{1}{2}\Delta t^2\mathcal{A}^2 + \cdots\right) q(x,0) \\ &= \left(I + \Delta t(\mathcal{A}+\mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A}^2 + 2\mathcal{B}\mathcal{A} + \mathcal{B}^2) + \cdots\right) q(x,0). \end{aligned}$$

In true solution operator,

$$\begin{aligned} (\mathcal{A}+\mathcal{B})^2 &= (\mathcal{A}+\mathcal{B})(\mathcal{A}+\mathcal{B}) \\ &= \mathcal{A}^2 + \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} + \mathcal{B}^2. \end{aligned}$$

Splitting error

$$
\begin{aligned}
q(x,\Delta t) - q^{**}(x,\Delta t) &= \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}} e^{\Delta t\mathcal{A}}\right) q(x,0) \\
&= \frac{1}{2}\Delta t^2(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})q(x,0) + O(\Delta t^3).
\end{aligned}
$$

There is a splitting error unless the two operators commute.

Splitting error

$$\begin{aligned} q(x,\Delta t) - q^{**}(x,\Delta t) &= \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right) q(x,0) \\ &= \frac{1}{2}\Delta t^2(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})q(x,0) + O(\Delta t^3). \end{aligned}$$

There is a splitting error unless the two operators commute.

No splitting error for constant coefficient advection:

$$\mathcal{A} = u\partial_x, \;\; \mathcal{B} = v\partial_y \quad \mathcal{A}\mathcal{B}q = \mathcal{B}\mathcal{A}q = uvq_{xy}$$

Splitting error

$$\begin{aligned} q(x,\Delta t) - q^{**}(x,\Delta t) &= \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right) q(x,0) \\ &= \frac{1}{2}\Delta t^2(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})q(x,0) + O(\Delta t^3). \end{aligned}$$

There is a splitting error unless the two operators commute.

No splitting error for constant coefficient advection:

$$\mathcal{A} = u\partial_x, \;\; \mathcal{B} = v\partial_y \quad \mathcal{A}\mathcal{B}q = \mathcal{B}\mathcal{A}q = uvq_{xy}$$

There is a splitting error if u, v are varying:

$$\begin{aligned} \mathcal{A}\mathcal{B}q = u(x,y)\partial_x\left(v(x,y)\partial_y q\right) = uvq_{xy} + uv_xq_y, \\ \mathcal{B}\mathcal{A}q = v(x,y)\partial_y\left(u(x,y)\partial_x q\right) = vuq_{xy} + vu_yq_x. \end{aligned}$$

Splitting error

$$\begin{aligned} q(x,\Delta t) - q^{**}(x,\Delta t) &= \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}} e^{\Delta t\mathcal{A}}\right) q(x,0) \\ &= \frac{1}{2}\Delta t^2(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}) q(x,0) + O(\Delta t^3). \end{aligned}$$

There is a splitting error unless the two operators commute.

No splitting error for constant coefficient advection:

$$\mathcal{A} = u\partial_x, \;\; \mathcal{B} = v\partial_y \quad \mathcal{A}\mathcal{B}q = \mathcal{B}\mathcal{A}q = uvq_{xy}$$

There is a splitting error if u, v are varying:

$$\begin{aligned} \mathcal{A}\mathcal{B}q &= u(x,y)\partial_x \left(v(x,y)\partial_y q\right) = uvq_{xy} + uv_x q_y, \\ \mathcal{B}\mathcal{A}q &= v(x,y)\partial_y \left(u(x,y)\partial_x q\right) = vuq_{xy} + vu_y q_x. \end{aligned}$$

There is a splitting error for acoustics since $ABq_{xy} \neq BAq_{xy}$.

Commuting operators

Note that if A and B are simultaneously diagonalizable,

$$A = R\Lambda R^{-1}, \qquad B = RMR^{-1},$$

then

$$AB = R\Lambda MR^{-1} = RM\Lambda R^{-1} = BA$$

Commuting operators

Note that if A and B are simultaneously diagonalizable,

$$A = R\Lambda R^{-1}, \qquad B = RMR^{-1},$$

then

$$AB = R\Lambda MR^{-1} = RM\Lambda R^{-1} = BA$$

and conversely, if two diagonalizable matrices commute, then they are simultaneously diagonalizable.

Commuting operators

Note that if A and B are simultaneously diagonalizable,

$$A = R\Lambda R^{-1}, \qquad B = RMR^{-1},$$

then

$$AB = R\Lambda MR^{-1} = RM\Lambda R^{-1} = BA$$

and conversely, if two diagonalizable matrices commute, then they are simultaneously diagonalizable.

So matrices arising from isotropic PDEs do not commute.

Splitting error

$$q(x, \Delta t) - q^{**}(x, \Delta t) = \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right) q(x, 0)$$

Combining 2 steps gives:

$$\begin{aligned} q^{**}(x, \Delta t) &= \left(I + \Delta t\mathcal{B} + \frac{1}{2}\Delta t^2\mathcal{B}^2 + \cdots\right)\left(I + \Delta t\mathcal{A} + \frac{1}{2}\Delta t^2\mathcal{A}^2 + \cdots\right) q(x, 0) \\ &= \left(I + \Delta t(\mathcal{A} + \mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A}^2 + 2\mathcal{B}\mathcal{A} + \mathcal{B}^2) + \cdots\right) q(x, 0). \end{aligned}$$

In true solution operator,

$$\begin{aligned} (\mathcal{A} + \mathcal{B})^2 &= (\mathcal{A} + \mathcal{B})(\mathcal{A} + \mathcal{B}) \\ &= \mathcal{A}^2 + \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} + \mathcal{B}^2. \end{aligned}$$

Strang splitting

Advance the PDE by time step Δt by...

- Time step $\Delta t/2$ on A-problem,
- Time step Δt on B-problem,
- Time step $\Delta t/2$ on A-problem.

Formally second order if each solution method is.

$$\left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\right)q(x,0) = O(\Delta t^3).$$

Strang splitting

Advance the PDE by time step Δt by...

- Time step $\Delta t/2$ on A-problem,
- Time step Δt on B-problem,
- Time step $\Delta t/2$ on A-problem.

Formally second order if each solution method is.

$$\left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\frac{1}{2}\Delta t\mathcal{A}} e^{\Delta t\mathcal{B}} e^{\frac{1}{2}\Delta t\mathcal{A}}\right) q(x,0) = O(\Delta t^3).$$

In practice often little difference from "first order Godunov splitting" since after N steps,

$$q^N = e^{\frac{1}{2}\Delta t\mathcal{A}} e^{\Delta t\mathcal{B}} e^{\frac{1}{2}\Delta t\mathcal{A}} \;\; e^{\frac{1}{2}\Delta t\mathcal{A}} e^{\Delta t\mathcal{B}} e^{\frac{1}{2}\Delta t\mathcal{A}} \;\; e^{\frac{1}{2}\Delta t\mathcal{A}} e^{\Delta t\mathcal{B}} e^{\frac{1}{2}\Delta t\mathcal{A}} \;\cdots$$
$$e^{\frac{1}{2}\Delta t\mathcal{A}} e^{\Delta t\mathcal{B}} e^{\frac{1}{2}\Delta t\mathcal{A}} \;\; e^{\frac{1}{2}\Delta t\mathcal{A}} e^{\Delta t\mathcal{B}} e^{\frac{1}{2}\Delta t\mathcal{A}} \; q^0$$

Strang splitting

Advance the PDE by time step Δt by...

- Time step $\Delta t/2$ on A-problem,
- Time step Δt on B-problem,
- Time step $\Delta t/2$ on A-problem.

Formally second order if each solution method is.

$$\left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\right) q(x,0) = O(\Delta t^3).$$

In practice often little difference from “first order Godunov splitting” since after N steps,

$$\begin{aligned} q^N = e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\ \ e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\ \ e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\ \cdots \\ e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\ \ e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\ q^0 \\ = e^{\frac{1}{2}\Delta t\mathcal{A}}\left(e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right)^N e^{\frac{1}{2}\Delta t\mathcal{A}}\ q^0 \end{aligned}$$

Example of splitting error for source term

Advection-reaction equation: $q_t + uq_x = -\beta(x)q$

Then

$$\frac{d}{dt}q(X(t),t) = -\beta(X(t))\,q(X(t),t) \qquad \text{(exponential decay)}$$

along characteristic $X(t) = x_0 + ut$.

Example of splitting error for source term

Advection-reaction equation: $q_t + uq_x = -\beta(x)q$

Then

$$\frac{d}{dt}q(X(t),t) = -\beta(X(t))\,q(X(t),t) \qquad \text{(exponential decay)}$$

along characteristic $X(t) = x_0 + ut$.

Splitting: Take $\mathcal{A} = -u\partial_x$ and $\mathcal{B} = -\beta(x)$.

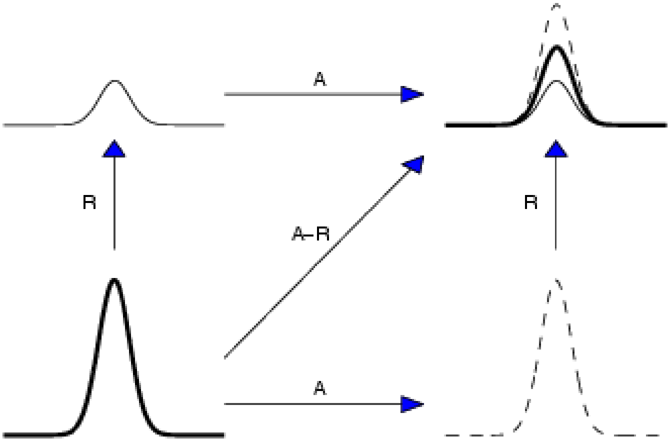
Then:

$$\begin{aligned} \mathcal{AB}q &= u\partial_x(\beta(x)q) = u\beta(x)q_x + u\beta'(x)q \\ \mathcal{BA}q &= \beta(x)uq_x \end{aligned}$$

Splitting error unless $\beta(x) =$ constant

Splitting error in advection-reaction (decay)

$q_t + uq_x = -\beta(x)q$ with $\beta(x)$ decreasing as x increases



β large β smaller

Taylor series in 2d for dimensional splitting

Consider $q_t + Aq_x + Bq_y = 0$.

$$q_{tt} = -Aq_{tx} - Bq_{ty} \quad = \; A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}$$

Taylor series in 2d for dimensional splitting

Consider $q_t + Aq_x + Bq_y = 0$.

$$q_{tt} = -Aq_{tx} - Bq_{ty} \quad = \; A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}$$

$$\begin{aligned} q(x,y,t+\Delta t) &= q + \Delta t\, q_t + \frac{1}{2}\Delta t^2 q_{tt} + \cdots \\ &= q - \Delta t(Aq_x + Bq_y) \\ &\quad + \frac{1}{2}\Delta t^2 \left[A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}\right] + \cdots \end{aligned}$$

Taylor series in 2d for dimensional splitting

Consider $q_t + Aq_x + Bq_y = 0$.

$$q_{tt} = -Aq_{tx} - Bq_{ty} \quad = \; A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}$$

$$\begin{aligned}
q(x, y, t + \Delta t) &= q + \Delta t\, q_t + \frac{1}{2}\Delta t^2 q_{tt} + \cdots \\
&= q - \Delta t(Aq_x + Bq_y) \\
&\quad + \frac{1}{2}\Delta t^2 \left[A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}\right] + \cdots \\
&= q - \Delta t\, Aq_x + \frac{1}{2}\Delta t^2 A^2 q_{xx} \\
&\quad - \Delta t\, Bq_y + \frac{1}{2}\Delta t^2 B^2 q_{yy} \\
&\quad + \frac{1}{2}\Delta t^2 [ABq_{yx} + BAq_{xy}] \quad \text{cross derivatives}
\end{aligned}$$

Dimensional splitting of upwind on $q_t + Aq_x + Bq_y = 0$

$$Q^*_{ij} = Q^n_{ij} - \frac{\Delta t}{\Delta x}[B^+(Q^n_{ij} - Q^n_{i,j-1}) + B^-(Q^n_{i,j+1} - Q^n_{ij})]$$

$$Q^{n+1}_{ij} = Q^*_{ij} - \frac{\Delta t}{\Delta x}[A^+(Q^*_{ij} - Q^*_{i-1,j}) + A^-(Q^*_{i+1,j} - Q^*_{ij})$$

Dimensional splitting of upwind on $q_t + Aq_x + Bq_y = 0$

$$Q^*_{ij} = Q^n_{ij} - \frac{\Delta t}{\Delta x}[B^+(Q^n_{ij} - Q^n_{i,j-1}) + B^-(Q^n_{i,j+1} - Q^n_{ij})]$$

$$Q^{n+1}_{ij} = Q^*_{ij} - \frac{\Delta t}{\Delta x}[A^+(Q^*_{ij} - Q^*_{i-1,j}) + A^-(Q^*_{i+1,j} - Q^*_{ij})$$

Consider one term, e.g. the one in blue above

$$\begin{aligned}\frac{\Delta t}{\Delta x}A^+(Q^*_{ij} - Q^*_{i-1,j}) &= \frac{\Delta t}{\Delta x}A^+\left[Q^n_{ij} - \frac{\Delta t}{\Delta x}\left(B^+(Q^n_{ij} - Q^n_{i,j-1}) + B^-(Q^n_{i,j+1} - Q^n_{ij})\right)\right] \\ &\quad - A^+\left[Q^n_{i-1,j} - \frac{\Delta t}{\Delta x}\left(B^+(Q^n_{i-1,j} - Q^n_{i-1,j-1}) + B^-(Q^n_{i-1,j+1} - Q^n_{i-1,j})\right)\right]\end{aligned}$$

Dimensional splitting of upwind on $q_t + Aq_x + Bq_y = 0$

$$Q^*_{ij} = Q^n_{ij} - \frac{\Delta t}{\Delta x}[B^+(Q^n_{ij} - Q^n_{i,j-1}) + B^-(Q^n_{i,j+1} - Q^n_{ij})]$$

$$Q^{n+1}_{ij} = Q^*_{ij} - \frac{\Delta t}{\Delta x}[A^+(Q^*_{ij} - Q^*_{i-1,j}) + A^-(Q^*_{i+1,j} - Q^*_{ij})$$

Consider one term, e.g. the one in blue above

$$\begin{aligned}\frac{\Delta t}{\Delta x}A^+(Q^*_{ij} - Q^*_{i-1,j}) = \frac{\Delta t}{\Delta x}A^+ &\left[Q^n_{ij} - \frac{\Delta t}{\Delta x}\left(B^+(Q^n_{ij} - Q^n_{i,j-1}) + B^-(Q^n_{i,j+1} - Q^n_{ij})\right)\right] \\ - A^+ &\left[Q^n_{i-1,j} - \frac{\Delta t}{\Delta x}\left(B^+(Q^n_{i-1,j} - Q^n_{i-1,j-1}) + B^-(Q^n_{i-1,j+1} - Q^n_{i-1,j})\right)\right]\end{aligned}$$

Includes, e.g.:

$$(\tfrac{\Delta t}{\Delta x})^2 A^+B^-(Q^n_{i,j+1} - Q^n_{ij} - Q^n_{i-1,j+1} + Q^n_{i-1,j}) \approx \tfrac{\Delta t^2 \Delta y}{\Delta x \Delta y} A^+B^- q_{xy}(x_i, y_j)$$

Upwind splitting of matrix product

In 1D, the upwind method is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)]$$

where

$$A = R\Lambda R^{-1} = R\Lambda^+ R^{-1} + R\Lambda^- R^{-1} = A^+ \ + \ A^-$$

Upwind splitting of matrix product

In 1D, the upwind method is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)]$$

where

$$A = R\Lambda R^{-1} = R\Lambda^+ R^{-1} + R\Lambda^- R^{-1} = A^+ \ + \ A^-$$

In 2D the unsplit generalization uses

$$AB = (A^+ + A^-)(B^+ + B^-) = A^+B^+ \ + \ A^+B^- \ + \ A^-B^+ \ + \ B^-A^-,$$
$$BA = (B^+ + A^-)(B^+ + A^-) = B^+A^+ \ + \ B^+A^- \ + \ B^-A^+ \ + \ B^-A^-.$$

Upwind splitting of matrix product

In 1D, the upwind method is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)]$$

where

$$A = R\Lambda R^{-1} = R\Lambda^+ R^{-1} + R\Lambda^- R^{-1} = A^+ \ + \ A^-$$

In 2D the unsplit generalization uses

$$AB = (A^+ + A^-)(B^+ + B^-) = A^+B^+ \ + \ A^+B^- \ + \ A^-B^+ \ + \ B^-A^-,$$
$$BA = (B^+ + A^-)(B^+ + A^-) = B^+A^+ \ + \ B^+A^- \ + \ B^-A^+ \ + \ B^-A^-.$$

Scalar advection: only one term is nonzero in each product,
e.g. $u > 0, \ v < 0 \implies uv = vu = u^+v^-$