Finite Volume Methods for Hyperbolic Problems

Approximate Riemann Solvers

- HLL method
- Linearized Jacobian approach
- Roe solvers
- Shallow water example
- HLLE method and positivity
- Harten-Hyman entropy fix

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html

In particular see: **Shallow_water_approximate.ipynb**

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For flux-differencing methods: Compute approximation to flux at interface between cells.

Obtain high resolution via higher-order time stepping with flux limiter.

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For wave-propagation algorithm: Approximate true Riemann solution by set of waves consisting of finite jumps propagating at constant speeds.

Can then apply high-resolution wave limiters.

May require entropy fix if a wave should be transonic rarefaction.

Wave propagation methods

- Solving Riemann problem gives waves $\mathcal{W}^p_{i-1/2}$,

$$Q_i - Q_{i-1} = \sum_p \mathcal{W}^p_{i-1/2}$$

and speeds $s^p_{i-1/2}$. (Usually approximate solver used.)

- These waves update neighboring cell averages depending on sign of s^p (Godunov's method) via fluctuations.

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- Waves also give (characteristic) decomposition of slopes:

$$q_x(x_{i-1/2}, t) \approx \frac{Q_i - Q_{i-1}}{\Delta x} = \frac{1}{\Delta x} \sum_p \mathcal{W}^p_{i-1/2}$$

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- Apply limiter to each wave to obtain $\widetilde{\mathcal{W}}^p_{i-1/2}$.
- Use limited waves in second-order correction terms.

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Some approaches to approximating Riemann solution
by a set of jump discontinuities:

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Local linearization: Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A} q_x = 0, \qquad \text{where } \hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave}).$$

Eigenvectors give waves. Roe solver $\Longrightarrow$ conservative

HLL Solver

Harten – Lax – van Leer (1983): Given $Q_\ell, Q_r \in \mathbb{R}^m$ for $m \geq 2$,
Use only 2 waves with a single interediate state Q^*.

$s^1 \approx$ minimum characteristic speed
$s^2 \approx$ maximum characteristic speed

$$\mathcal{W}^1 = Q^* - Q_\ell, \qquad \mathcal{W}^2 = Q_r - Q^*$$

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$$\implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}.$$

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Choice of speeds:

- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of $f'(Q_\ell)$ and $f'(Q_r)$.

HLL Solver for Shallow Water Equations

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

Choose e.g.

$$s^1 = u_\ell - \sqrt{gh_\ell},$$

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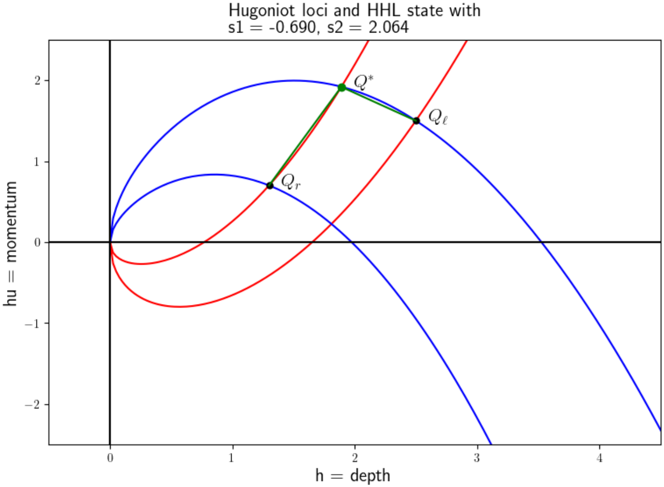
Then

$$Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}$$
$$= \frac{1}{s^1 - s^2}\left[\begin{array}{c} h_r u_r - h_\ell u_\ell - s^2 h_r + s^1 h_\ell \\ \left(h_r u_r^2 + \frac{1}{2}gh_r^2\right) - \left(h_\ell u_\ell^2 + \frac{1}{2}gh_\ell^2\right) - s^2 h_r u_r + s^1 h_\ell u_\ell \end{array}\right]$$

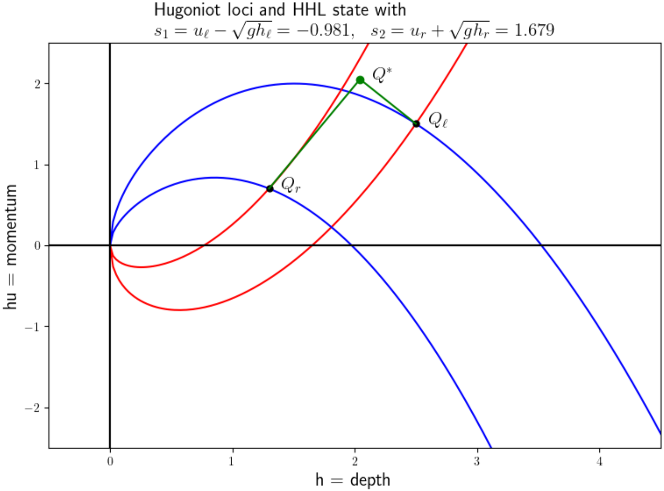
HLL solver for shallow water

If we use the shock speeds from the exact two-shock solution, looks perfect:



HLL solver for shallow water

Using $s_1 = \lambda^1(q_\ell)$ and $s_2 = \lambda^2(q_r)$:



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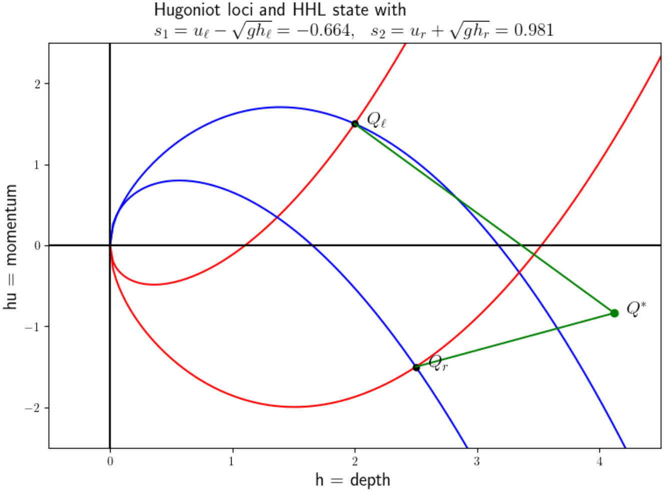
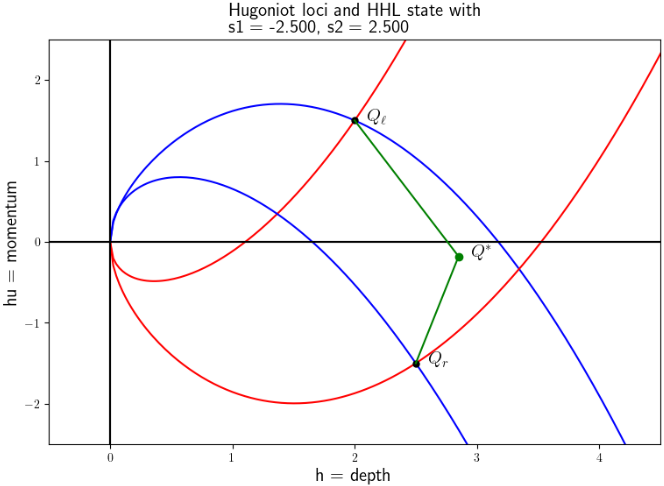


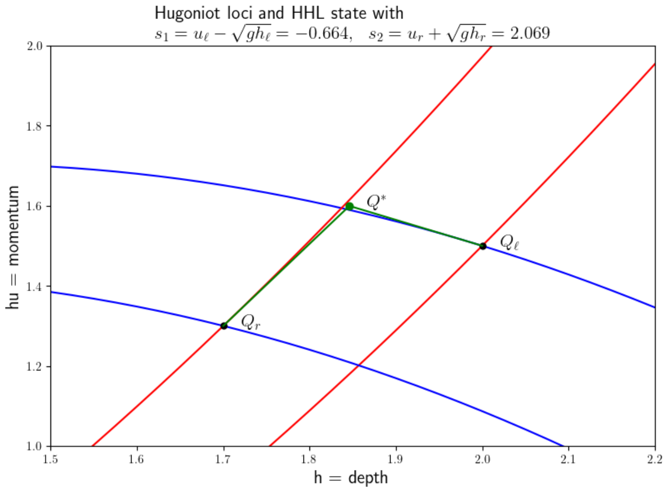
HLL solver for shallow water

Using different choice of $s_1, \ s_2$:



HLL solver for shallow water

If $\Delta Q = Q_r - Q_\ell$ is small, then eigenvalues nearly constant.
For smooth flow, HLL is very accurate (for $m = 2$).



Approximate Riemann Solvers — Local Linearization

Approximate true Riemann solution by set of waves consisting of finite jumps propagating at constant speeds.

Local linearization:

Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A} q_x = 0,$$

where $\hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave})$.

Then decompose

$$q_r - q_l = \alpha^1 \hat{r}^1 + \cdots \alpha^m \hat{r}^m$$

to obtain waves $\mathcal{W}^p = \alpha^p \hat{r}^p$ with speeds $s^p = \hat{\lambda}^p$.

Approximate Riemann Solvers

How to use?

One approach: determine $Q^* =$ state along $x/t = 0$,

$$Q^* = Q_{i-1} + \sum_{p:s^p<0} \mathcal{W}^p, \quad F_{i-1/2} = f(Q^*),$$

$$\mathcal{A}^-\Delta Q = F_{i-1/2} - f(Q_{i-1}), \quad \mathcal{A}^+\Delta Q = f(Q_i) - F_{i-1/2}.$$

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Or, sometimes can use:

$$\mathcal{A}^-\Delta Q = \sum_{p:s^p<0} s^p\mathcal{W}^p, \qquad \mathcal{A}^+\Delta Q = \sum_{p:s^p>0} s^p\mathcal{W}^p.$$

Conservative only if $\mathcal{A}^-\Delta Q + \mathcal{A}^+\Delta Q = f(Q_i) - f(Q_{i-1})$.

This holds for Roe solver.

Roe Solver

Given q_ℓ, q_r, solve $q_t + \hat{A} q_x = 0$ where $\hat{A}$ chosen to satisfy

$$\hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

Then:

- Good approximation for weak waves (smooth flow)

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- Single shock captured exactly:

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$$\mathcal{A}^- \Delta Q_{i-1/2} + \mathcal{A}^+ \Delta Q_{i-1/2} = \sum s^p_{i-1/2} \mathcal{W}^p_{i-1/2} = \hat{A} \sum \mathcal{W}^p_{i-1/2}$$
$$= \hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Roe solver for Shallow Water

Given $h_\ell,\ u_\ell,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_\ell + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_\ell}u_\ell + \sqrt{h_r}u_r}{\sqrt{h_\ell} + \sqrt{hr}}$$

Then

$$\hat{A} = \text{Jacobian matrix evaluated at this average state}$$

satisfies

$$\hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.

Roe solver for Shallow Water

Given $h_l,\ u_l,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{hr}}$$

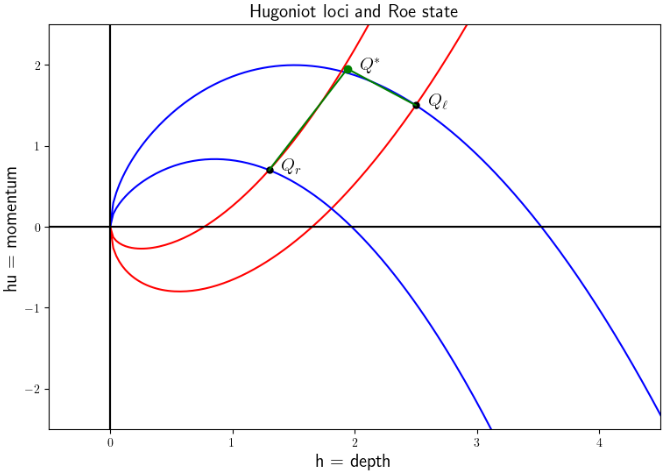
Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}^1 = \hat{u} - \hat{c}, \quad \hat{\lambda}^2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{g\bar{h}}.$$

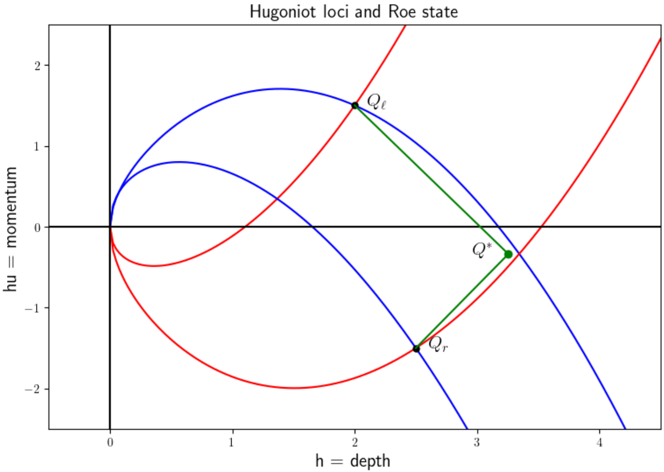
Eigenvectors:

$$\hat{r}^1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \qquad \hat{r}^2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$

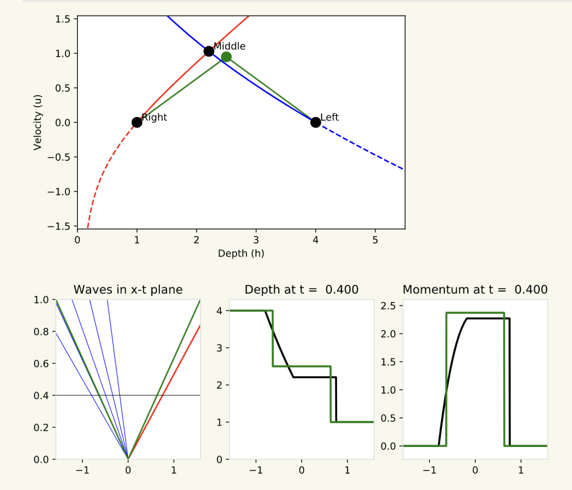
Roe solver for shallow water



Roe solver for shallow water



Dam break problem with Roe solver

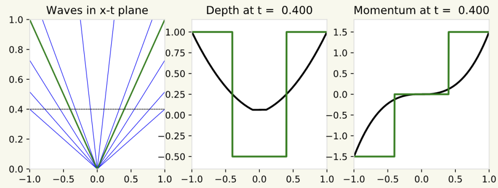


Note that rarefaction replaced by jump.

Example from **RpJs**

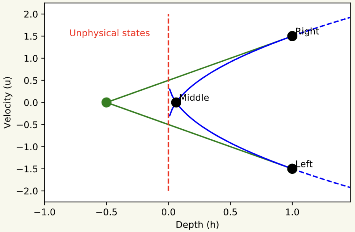
Widgets can be used to experiment.

Nonphyiscal solution with Roe solver



In [12]:
```
sw.phase_plane_plot(q_l,q_r,g=1.,y_axis='u',
                    approx_states=states, hmin = -1)
plt.plot([0,0], [-2,2], 'r--')
plt.text(-0.8,1.5,'Unphysical states',color='red')
```

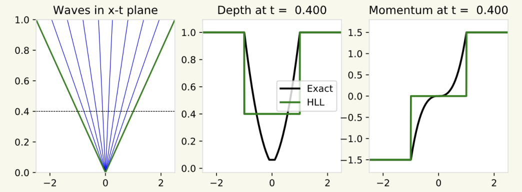
Out[12]: `Text(-0.8,1.5,'Unphysical states')`



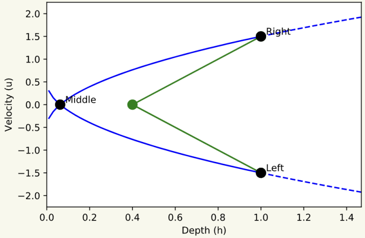
For data that gives near dry state in Q_m, Roe solver may give negative depth.

Example from **RpJs**

HLLE preserves positivity



In [25]:
```
sw.phase_plane_plot(q_l,q_r,g=1.,y_axis='u',
                    approx_states=states['hll'])
```



For data that gives near dry state in Q_m, Roe solver may give negative depth.

Choosing s_1, s_2 as characteristic speeds in HLL does much better in this case.

Example from **RpJs**

HLLE Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Based on characteristic speeds and Roe averages:

$$s^1_{i-1/2} = \min_p(\min(\lambda^p_i, \hat{\lambda}^p_{i-1/2})),$$
$$s^2_{i-1/2} = \max_p(\max(\lambda^p_{i+1}, \hat{\lambda}^p_{i-1/2})).$$

where

λ^p_i is the pth eigenvalue of the Jacobian $f'(Q_i)$,

$\hat{\lambda}^p_{i-1/2}$ is the pth eigenvalue using Roe average $f'(\hat{Q}_{i-1/2})$

HLLE Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Can also show that:

- If Riemann solution consists of single shock,
  then Roe speed is used $\Longrightarrow$ exact solution in this case.

HLLE Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Can also show that:

- If Riemann solution consists of single shock,
  then Roe speed is used $\Longrightarrow$ exact solution in this case.
- No entropy fix needed.
  (More diffusive than Roe solver.)

Harten-Hyman entropy fix

For any wave splitting $Q_i - Q_{i-1} = \sum \mathcal{W}^p$, with speeds $\hat{\lambda}^p$.

Define

$$q_\ell^k = Q_{i-1} + \sum_{p=1}^{k-1} \mathcal{W}^p, \qquad q_r^k = q_\ell^k + \mathcal{W}^k$$

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Define

$$q_\ell^k = Q_{i-1} + \sum_{p=1}^{k-1} \mathcal{W}^p, \qquad q_r^k = q_\ell^k + \mathcal{W}^k$$

If $\lambda_\ell^k \equiv \lambda^k(q_\ell^k) < 0 < \lambda^k(q_r^k) \equiv \lambda_r^k$ then replace $\mathcal{W}^k$ by

$$\mathcal{W}_\ell^k = \beta \mathcal{W}^k, \qquad \text{speed} = \lambda_\ell^k < 0,$$
$$\mathcal{W}_r^k = (1-\beta)\mathcal{W}^k, \qquad \text{speed} = \lambda_r^k > 0.$$

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$$\begin{aligned} \mathcal{W}_\ell^k &= \beta \mathcal{W}^k, & \text{speed} &= \lambda_\ell^k < 0, \\ \mathcal{W}_r^k &= (1-\beta)\mathcal{W}^k, & \text{speed} &= \lambda_r^k > 0. \end{aligned}$$

Conservation requires:

$$\lambda_\ell^k \mathcal{W}_\ell^k + \lambda_r^k \mathcal{W}_r^k = \hat{\lambda}^k \mathcal{W}^k, \qquad \Longrightarrow \beta = \frac{\lambda_r^k - \hat{\lambda}^k}{\lambda_r^k - \lambda_\ell^k}$$

Harten-Hyman entropy fix

In wave propagation algorithm, leave $\mathcal{W}^k$ alone for high-resolution correction terms (with limiters).

Similar to entopy fix for scalar problem:

Only need to modify the fluctuations in the “Godunov update”

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Only need to modify the fluctuations in the "Godunov update"

$$\mathcal{A}^-\Delta Q = \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}^p, \qquad \mathcal{A}^+\Delta Q = \sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}^p,$$

Usually $\quad (\lambda^p)^- = \min(\lambda^p, 0), \qquad (\lambda^p)^+ = \max(\lambda^p, 0)$.

Modify for field k:

$$(\lambda^k)^- = \beta\lambda_\ell^k < 0, \qquad (\lambda^k)^+ = (1-\beta)\lambda_r^k > 0,$$

so that

$$(\lambda^k)^-\mathcal{W}^k = \lambda_\ell^k\beta\mathcal{W}^k \qquad (\lambda^k)^+\mathcal{W}^k = \lambda_r^k(1-\beta)\mathcal{W}^k$$