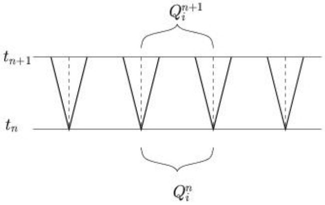
Finite Volume Methods for Nonlinear Systems

- Wave propagation method for systems
- High-resolution methods using wave limiters
- Example for shallow water equations

Godunov's Method for $q_t + f(q)_x = 0$



1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}^p_{i-1/2}$ and speeds $s^p_{i-1/2}$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

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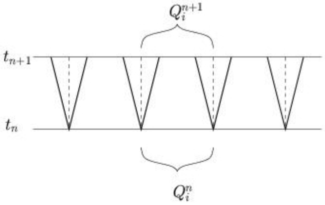

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1. Compute new cell averages by integrating over cell at t_{n+1},

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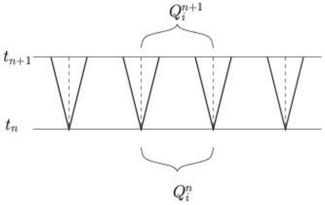

Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

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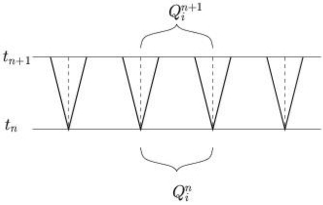

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3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

Approximate Riemann solver

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

For scalar advection $m = 1$, only one wave.

$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2} = Q_i - Q_{i-1}$ and $s_{i-1/2} = u$,

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For scalar nonlinear: Use same formulas with

$$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2}, \;\; s_{i-1/2} = (f(Q_i) - f(Q_{i-1}))/(Q_i - Q_{i-1}).$$

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Replacing rarefaction with shock: also exact (after averaging), except in case of transonic rarefaction.

Wave limiters for scalar nonlinear

For $q_t + f(q)_x = 0$, just one wave: $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$.

Godunov:

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"Lax-Wendroff":

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right] - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\mathcal{W}_{i-1/2}$$

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High-resolution method:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\widetilde{\mathcal{W}}_{i-1/2}$$

$\widetilde{\mathcal{W}}_{i-1/2} = \phi(\theta)\,\mathcal{W}_{i-1/2}$, where $\theta_{i-1/2} = \mathcal{W}_{I-1/2}/\mathcal{W}_{i-1/2}$.

Extension to constant coefficient linear systems

Approach 1: Diagonalize the system to

$$q_t + Aq_x \implies w_t + \Lambda w_x = 0, \quad q = Rw$$

$W^n = R^{-1}Q^n$, Apply scalar algorithm, Set $Q^{n+1} = RW^{n+1}$.

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Apply a wave limiter to each wave (comparing scalars $\alpha_{i-1/2}^p$).

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Note: Limiters are applied to waves or characteristic components, not to original variables.

Wave-propagation form of high-resolution method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(s_{i-1/2}^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(s_{i+1/2}^p)^-\mathcal{W}_{i+1/2}^p\right]$$
$$- \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

Correction flux:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{M_w}|s_{i-1/2}^p|\left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|\right)\widetilde{\mathcal{W}}_{i-1/2}^p$$

where $\widetilde{\mathcal{W}}_{i-1/2}^p$ is a limited version of $\mathcal{W}_{i-1/2}^p$ to avoid oscillations.

(Unlimited $\widetilde{\mathcal{W}}^p = \mathcal{W}^p \implies$ Lax-Wendroff for a linear system.)

Approximate Riemann Solvers

Some approaches to approximating Riemann solution
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Local linearization: Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A} q_x = 0, \qquad \text{where } \hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave}).$$

Eigenvectors give waves. Roe solver $\Longrightarrow$ conservative

Wave limiters for linear system

$Q_i - Q_{i-1}$ is split into waves $\mathcal{W}^p_{i-1/2} = \alpha^p_{i-1/2} r^p \in \mathbb{R}^m$.

For constant coefficient linear system: r^p is constant vector,
Only the scalar α^p varies.

Replace by $\widetilde{\mathcal{W}}^p_{i-1/2} = \Phi(\theta^p_{i-1/2})\mathcal{W}^p_{i-1/2}$ where

$$\theta^p_{i-1/2} = \frac{\alpha^p_{I-1/2}}{\alpha^p_{i-1/2}}$$

where

$$I = \begin{cases} i-1 & \text{if } s^p_{i-1/2} > 0 \\ i+1 & \text{if } s^p_{i-1/2} < 0. \end{cases}$$

In the scalar case this reduces to

$$\theta^1_{i-1/2} = \frac{\mathcal{W}^1_{I-1/2}}{\mathcal{W}^1_{i-1/2}} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

Wave limiters for system

$Q_i - Q_{i-1}$ is split into waves $\mathcal{W}^p_{i-1/2} \in \mathbb{R}^m$ with speeds $s^p_{i-1/2}$.
Upwind cell in family p:

$$I = \begin{cases} i-1 & \text{if } s^p_{i-1/2} > 0 \\ i+1 & \text{if } s^p_{i-1/2} < 0. \end{cases}$$

To compare $\mathcal{W}^p_{i-1/2}$ to $\mathcal{W}^p_{I-1/2}$ we want to reduce to a scalar
$\theta^p_{i-1/2} \approx 1$ where the solution is smooth,
negative near extreme points of this wave component.

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Use projection of $\mathcal{W}^p_{I-1/2}$ onto $\mathcal{W}^p_{i-1/2}$:

$$\left(\frac{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{I-1/2}}{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{i-1/2}}\right) \mathcal{W}^p_{i-1/2} \quad \text{compared to} \quad \mathcal{W}^p_{i-1/2}$$

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Ratio of coefficients: $\displaystyle \theta^p_{i-1/2} = \frac{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{I-1/2}}{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{i-1/2}}$

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Replace $\mathcal{W}^p_{i-1/2}$ by $\widetilde{\mathcal{W}}^p_{i-1/2} = \phi(\theta^p_{i-1/2})\mathcal{W}^p_{i-1/2}$. ($\phi(\theta)$ = limiter)

Wave limiters for system with eigendecomposition

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constant-coefficient:

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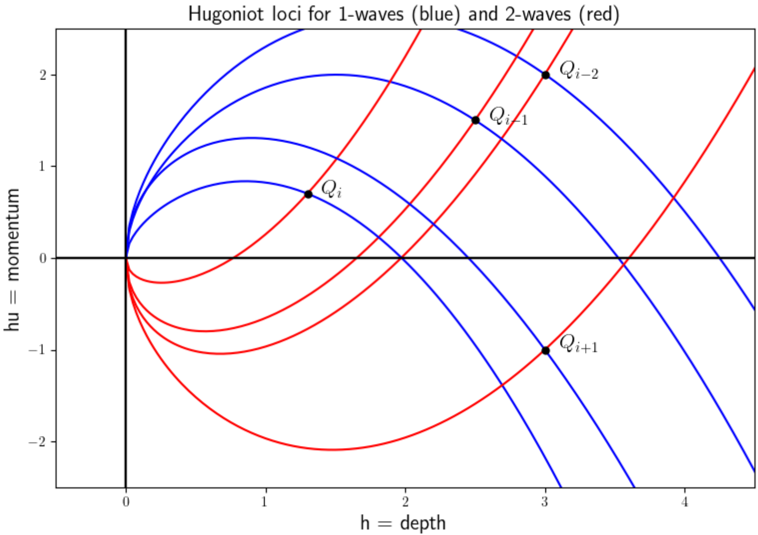
Limiters – shallow water equation



Note that speeds are $s = \Delta(hu)/\Delta(h) =$ slope between states.

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- Waves also give (characteristic) decomposition of slopes:

$$q_x(x_{i-1/2}, t) \approx \frac{Q_i - Q_{i-1}}{\Delta x} = \frac{1}{\Delta x} \sum_p \mathcal{W}^p_{i-1/2}$$

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- Apply limiter to each wave to obtain $\widetilde{\mathcal{W}}^p_{i-1/2}$.
- Use limited waves in second-order correction terms.