Finite Volume Methods for Hyperbolic Problems

# Convergence to Weak Solutions and Nonlinear Stability

- Lax-Wendroff Theorem
- Entropy consistent finite volume methods
- Nonlinear stability
- Total Variation stability

# Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \Delta t (F_{+\infty} - F_{-\infty}).$$

Note: an isolated shock must travel at the right speed!

$$\frac{\partial}{\partial t} \int_{x_1}^{x_2} q(x,t)\,dx = F(x_1) - F(x_2).$$

# Weak solutions to $q_t + f(q)_x = 0$

Alternatively, multiply PDE by smooth test function $\phi(x,t)$, with compact support ($\phi(x,t) \equiv 0$ for $|x|$ and $t$ sufficiently large), and then integrate over rectangle,

$$\int_0^\infty \int_{-\infty}^\infty \big(q_t + f(q)_x\big)\phi(x,t)\,dx\,dt$$

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$$\int_0^\infty \int_{-\infty}^\infty \big(q_t + f(q)_x\big)\phi(x,t)\,dx\,dt$$

Then we can integrate by parts to get

$$\int_0^\infty \int_{-\infty}^\infty \big(q\phi_t + f(q)\phi_x\big)\,dx\,dt = -\int_{-\infty}^\infty q(x,0)\phi(x,0)\,dx.$$

$q(x,t)$ is a weak solution if this holds for all such $\phi$.

# Lax-Wendroff Theorem

Suppose the method is conservative and consistent with $q_t + f(q)_x = 0$,

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x, t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges (need stability).

Two sequences might converge to different weak solutions.

Also need to satisfy an entropy condition.

# Sketch of proof of Lax-Wendroff Theorem

Conservative numerical method:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

Multiply by $\Phi_i^n$:    (cell-averaged version of test function $\phi(x,t)$)

$$\Phi_i^n Q_i^{n+1} = \Phi_i^n Q_i^n - \frac{\Delta t}{\Delta x}\Phi_i^n(F_{i+1/2}^n - F_{i-1/2}^n).$$

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This is true for all values of $i$ and $n$ on each grid.
Now sum over all $i$ and $n \geq 0$ to obtain

$$\sum_{n=0}^{\infty}\sum_{i=-\infty}^{\infty}\Phi_i^n(Q_i^{n+1} - Q_i^n) = -\frac{\Delta t}{\Delta x}\sum_{n=0}^{\infty}\sum_{i=-\infty}^{\infty}\Phi_i^n(F_{i+1/2}^n - F_{i-1/2}^n).$$

Use summation by parts to transfer differences to $\Phi$ terms.

# Summation by parts

Integration by parts:

$$\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx.$$

Consider sum:

$$\sum_{i=1}^{N} u_i(v_i - v_{i-1})$$

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$$= u_1(v_1 - v_0) + u_2(v_2 - v_1) + \cdots$$

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&\qquad + u_{N-1}(v_{N-1} - v_{N-2}) + u_N(v_N - v_{N-1}) \\
&= -u_1 v_0 - (u_2 - u_1)v_1 - (u_3 - u_2)v_2 + \cdots \\
&\qquad - (u_N - u_{N-1})v_{N-1} + u_N v_N
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&= u_{N-1}v_{N-1} - u_1 v_0 - \sum_{i=1}^{N-1} (u_{i+1} - u_i)v_i
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Obtain analog of weak form of conservation law:

$$\Delta x \Delta t \left[ \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} \left( \frac{\Phi_i^n - \Phi_i^{n-1}}{\Delta t} \right) Q_i^n \right.$$
$$\left. + \sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \left( \frac{\Phi_{i+1}^n - \Phi_i^n}{\Delta x} \right) F_{i-1/2}^n \right] = -\Delta x \sum_{i=-\infty}^{\infty} \Phi_i^0 Q_i^0.$$

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Can show that any limiting function

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Must use $F_{i-1/2}^n \to f(Q_i^n)$ almost everywhere, using consistency of numerical flux $F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i)$.

# Analog of Lax-Wendroff proof for entropy

Suppose the numerical flux function $\mathcal{F}(Q_{i-1}, Q_i)$ leads to a numerical entropy flux $\Psi(Q_{i-1}, Q_i)$

such that the following discrete entropy inequality holds:

$$\eta(Q_i^{n+1}) \le \eta(Q_i^n) - \frac{\Delta t}{\Delta x}\left[\Psi^n_{i+1/2} - \Psi^n_{i-1/2}\right].$$

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Then multiply by test function $\Phi_i^n$, sum and use summation by parts to get discrete form of integral form of entropy condition.

$\Longrightarrow$ If numerical approximations converge to some function, then the limiting function satisfies the entropy condition.

# Entropy consistency of Godunov's method

For Godunov's method, $F(Q_{i-1}, Q_i) = f(Q^{\Downarrow}_{i-1/2})$

where $Q^{\Downarrow}_{i-1/2}$ is the constant value along $x_{i-1/2}$ in the Riemann solution.

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If we use exact solution satisfying the entropy condition, then

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta\big(\tilde{q}^n(x,t_{n+1})\big)\,dx \;\;\le\;\; \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta\big(\tilde{q}^n(x,t_n)\big)\,dx$$

$$+\frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \psi(\tilde{q}^n(x_{i-1/2},t)\,dt - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \psi(\tilde{q}^n(x_{i+1/2},t)\,dt$$

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$$= \eta(Q^n_i) - \frac{\Delta t}{\Delta x}(\Psi^n_{i+1/2} - \Psi^n_{i-1/2})$$

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Follows from Jensen's inequality for convex functions:

If $\eta''(q) \geq 0$ then The value of $\eta(q(x))$ evaluated at the average value of $q(x)$ is less than or equal to the average value of $\eta(q(x))$, i.e.,

$$\eta\left(\int q(x)\,dx\right) \leq \int \eta(q(x))\,dx.$$

# Convergence and stability

Let $q^n$ be cell averages of exact solution at time $t_n$

$$Q^n = q^n + E^n.$$

We apply the numerical method to obtain $Q^{n+1}$:

$$Q^{n+1} = \mathcal{N}(Q^n) = \mathcal{N}(q^n + E^n)$$

and the global error is now

$$\begin{aligned}
E^{n+1} &= Q^{n+1} - q^{n+1} \\
&= \mathcal{N}(q^n + E^n) - q^{n+1} \\
&= \mathcal{N}(q^n + E^n) - \mathcal{N}(q^n) + \mathcal{N}(q^n) - q^{n+1} \\
&= [\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)] + \Delta t\, \tau^n.
\end{aligned}$$

where $\tau^n$ is the local trucation error introduced in this step.

# Convergence and stability

$$E^{n+1} = [\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)] + \Delta t\, \tau^n.$$

so

$$\|E^{n+1}\| \leq \|\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)\| + \Delta t\, \|\tau^n\|$$

If

$$\|\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)\| \leq \|E^n\|$$

then

$$\begin{aligned} \|E^N\| &\leq \|E^0\| + \Delta t \sum_{n=1}^{N-1} \|\tau\| \\ &\leq (\|E^0\| + T\|\tau\|) \qquad \text{(for } N\Delta t = T\text{).} \end{aligned}$$

# Nonlinear stability

Would like to show

$$\|\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)\| \leq \|E^n\|$$

If method is linear, $\mathcal{N}(q^n + E^n) = \mathcal{N}(q^n) + \mathcal{N}(E^n)$, then enough to show:

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But in nonlinear case we need contractivity,

$$\|\mathcal{N}(P) - \mathcal{N}(Q)\| \le \|P - Q\|.$$

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Kružkov's Theorem (1970): Entropy stability for $\eta(q) = |q-k|$,

$$(|q-k|)_t + ((f(q)-f(k))\text{sgn}(q-k))_x \le 0$$

for all constants $k$ implies

$$\|q(\cdot,t) - w(\cdot,t)\|_1 \le \|q(\cdot,0) - w(\cdot,0)\|_1$$

for all $t \ge 0$. (1-norm contractivity)

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for all $t \geq 0$. (1-norm contractivity)

Numerical methods with this property are at best first order.

# TV Stability

A numerical method is Total Variation Bounded (TVB) if

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Function space BV: A set of functions such as

$$\{v \in L_1 : \ TV(v) \le R \text{ and } \text{Supp}(v) \subset [-M, M]\}$$

is a compact set, so any sequence of functions has a convergent subsequence.

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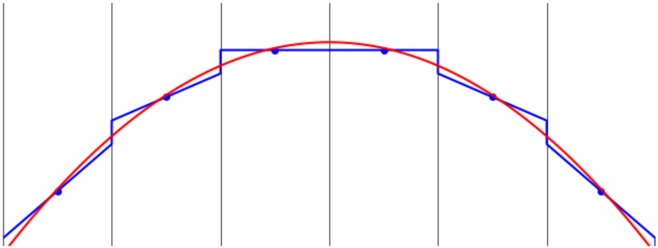
But then Lax-Wendroff Theorem $\implies q$ is a weak solution.
Contradiction.

# Accuracy at local extrema

TVD methods must clip local extrema:



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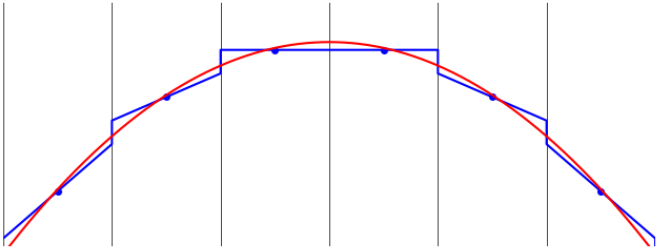

Clipping by $\Delta x^2$ each time step can lead to $\mathcal{O}(\Delta x)$ global errors

TVB methods: Only require $TV(Q^{n+1}) \leq (1 + \Delta t) TV(Q^n)$.

Essentially nonoscillatory (ENO) methods

# Order of accuracy isn't everything

Order $s$: $\|E^N\| = C(\Delta x)^s +$ higher order terms.

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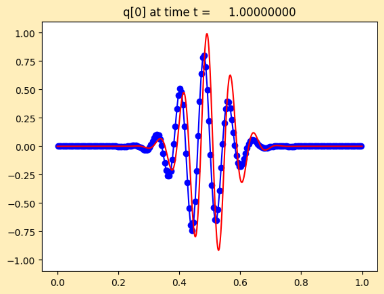
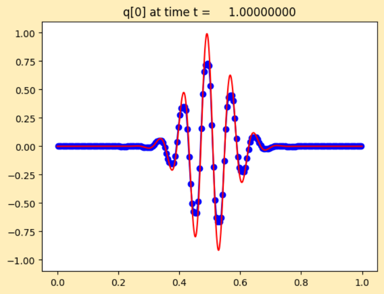
Comparison of Lax-Wendroff and a high-resolution method on linear advection equation with smooth wave packet data.



# Order of accuracy isn't everything

Order $s$: $\|E^N\| = C(\Delta x)^s +$ higher order terms.

Comparison of Lax-Wendroff and a high-resolution method on linear advection equation with smooth wave packet data.

The high-resolution method is not formally second-order accurate, but is more accurate on realistic grids.

Crossover in the max-norm is at $2800$ grid points.