Finite Volume Methods for Hyperbolic Problems

Admissible Solutions and Entropy Functions

- Weak solutions and conservation form
- Admissibility / entropy conditions
- Entropy functions
- Weak form of entropy condition
- Relation to vanishing viscosity solution

Weak solutions to $q_t + f(q)_x = 0$

$q(x,t)$ is a weak solution if it satisfies the integral form of the conservation law over all rectangles in space-time,

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx = \int_{t_1}^{t_2} f(q(x_1,t))\,dt - \int_{t_1}^{t_2} f(q(x_2,t))\,dt$$

Obtained by integrating

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

from t_n to t_{n+1}.

Weak solutions to $q_t + f(q)_x = 0$

Alternatively, multiply PDE by smooth test function $\phi(x,t)$, with compact support ($\phi(x,t) \equiv 0$ for $|x|$ and t sufficiently large), and then integrate over rectangle,

$$\int_0^\infty \int_{-\infty}^\infty \left(q_t + f(q)_x\right)\phi(x,t)\,dx\,dt$$

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Then we can integrate by parts to get

$$\int_0^\infty \int_{-\infty}^\infty \big(q\phi_t + f(q)\phi_x\big)\,dx\,dt = -\int_{-\infty}^\infty q(x,0)\phi(x,0)\,dx.$$

$q(x,t)$ is a weak solution if this holds for all such ϕ.

Weak solutions to $q_t + f(q)_x = 0$

A function $q(x,t)$ that is piecewise smooth with jump discontinuities is a weak solution only if:

- The PDE is satisfied where q is smooth,
- The jump discontinuities all satisfy the Rankine-Hugoniot conditions.

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Other admissibility conditions needed to pick out the physically correct weak solution, e.g.

- Vanishing viscosity limit,
- “Entropy conditions”

Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \Delta t (F_{+\infty} - F_{-\infty}).$$

Note: an isolated shock must travel at the right speed!

$$\frac{\partial}{\partial t} \int_{x_1}^{x_2} q(x,t)\, dx = F(x_1) - F(x_2).$$

Nonlinear scalar conservation laws

Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = 0$.

Quasilinear form: $u_t + uu_x = 0$.

These are equivalent for smooth solutions, not for shocks!

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Upwind methods for $u > 0$:

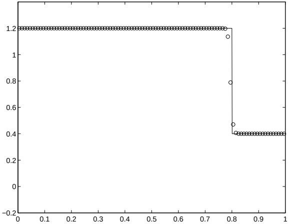
Conservative: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}((U_i^n)^2 - (U_{i-1}^n)^2)\right)$

Quasilinear: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n)$.

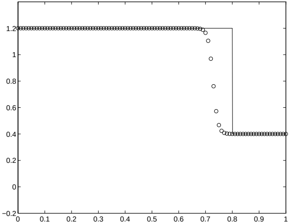
Ok for smooth solutions, not for shocks!

Importance of conservation form

Solution to Burgers' equation using conservative upwind:



Solution to Burgers' equation using quasilinear upwind:



Weak solutions depend on the conservation law

The conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0$$

both have the same quasilinear form

$$u_t + uu_x = 0$$

but have different weak solutions,

different shock speeds!

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$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad \Longrightarrow \quad s = \frac{1}{2}\frac{u_r^2 - u_\ell^2}{u_r - u_l} = \frac{1}{2}(u_\ell + u_r).$$

whereas

$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0 \quad \Longrightarrow \quad s = \frac{2}{3}\frac{u_r^3 - u_\ell^3}{u_r^2 - u_\ell^2}.$$

Speeds are different in general $\Longrightarrow$ different weak solutions.

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with $q_t + f(q)_x = 0$,

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x,t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges (need stability).

Two sequences might converge to different weak solutions.

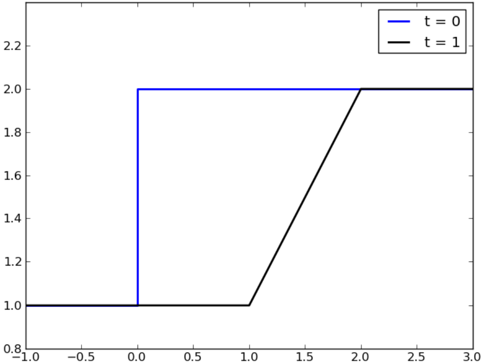
Also need to satisfy an entropy condition.

Weak solutions to Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \;\; u_r = 2$$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

"Physically correct" rarefaction wave solution:



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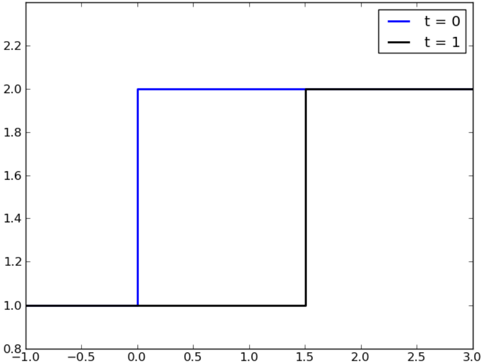
Entropy violating weak solution:



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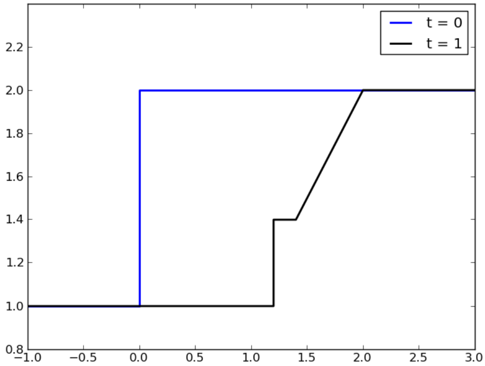
Another Entropy violating weak solution:



Vanishing viscosity solution

We want $q(x,t)$ to be the limit as $\epsilon \to 0$ of solution to

$$q_t + f(q)_x = \epsilon q_{xx}.$$

This selects a unique weak solution:

- Shock if $f'(q_l) > f'(q_r)$,
- Rarefaction if $f'(q_l) < f'(q_r)$.

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Lax Entropy Condition:

A discontinuity propagating with speed s in the solution of a convex scalar conservation law is admissible only if $f'(q_\ell) > s > f'(q_r)$, where $s = (f(q_r) - f(q_\ell))/(q_r - q_\ell)$.

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Note: This means characteristics must approach shock from both sides as t advances, not move away from shock!

Entropy (admissibility) conditions

We generally require additional conditions on a weak solution to a conservation law, to pick out the unique solution that is physically relevant.

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Entropy functions: Function of q that "behaves like" physical entropy for the conservation law being studied.

NOTE: Mathematical entropy functions generally chosen to
decrease for admissible solutions,
increase for entropy-violating solutions.

Entropy functions for convex scalar problems

Entropy function: $\eta : \mathbb{R} \to \mathbb{R}$ Entropy flux: $\psi : \mathbb{R} \to \mathbb{R}$

chosen so that $\eta(q)$ is convex ($\eta''(q) > 0$) (not < 0) and:

- $\eta(q)$ is conserved wherever the solution is smooth,

$$\eta(q)_t + \psi(q)_x = 0.$$

- Entropy decreases across an admissible shock wave.

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with equality where solution is smooth. $\eta(q)_t + \psi(q)_t \le 0$

Entropy functions

How to find η and ψ satisfying this?

$$\eta(q)_t + \psi(q)_x = 0$$

For smooth solutions gives

$$\eta'(q)q_t + \psi'(q)q_x = 0.$$

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Example: Burgers' equation, $f'(u) = u$ and take $\eta(u) = u^2$.

Then $\psi'(u) = 2u^2 \implies$ Entropy function: $\psi(u) = \frac{2}{3}u^3$.

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whereas

$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0 \quad \Longrightarrow \quad s^* = \frac{2}{3}\frac{u_r^3 - u_\ell^3}{u_r^2 - u_\ell^2}.$$

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If $u_\ell > u_r$ (correct shock) then $\frac{\partial}{\partial t}\int u^2\,dx < \psi(u_r^2) - \psi(u_\ell^2)$

If $u_\ell < u_r$ (entropy-violating) then $\frac{\partial}{\partial t}\int u^2\,dx > \psi(u_r^2) - \psi(u_\ell^2)$

Entropy condition for LWR traffic equations

$f(q) = q(1-q)$. Note that $q_t + (1-2q)q_x = 0$ where smooth

Again take entropy function $\eta(q) = q^2$ (we need $\eta''(q) > 0$)
Determine entropy flux by solving

$$\psi'(q) = \eta'(q)f'(q) = 2q(1-2q) \quad \Longrightarrow \quad \psi(q) = q^2 - \frac{4}{3}q^3.$$

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which $= 0$ from the original equation.
Why ≤ 0 for correct shock?   Consider vanishing viscosity...

Entropy functions

$$\int_{x_1}^{x_2} \eta(q(x,t_2))\,dx \le \int_{x_1}^{x_2} \eta(q(x,t_1))\,dx + \int_{t_1}^{t_2} \psi(q(x_1,t))\,dt - \int_{t_1}^{t_2} \psi(q(x_2,t))\,dt$$

comes from considering the vanishing viscosity solution:

$$q_t^\epsilon + f(q^\epsilon)_x = \epsilon q_{xx}^\epsilon \qquad \text{with } \epsilon > 0$$

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comes from considering the vanishing viscosity solution:

$$q^\epsilon_t + f(q^\epsilon)_x = \epsilon q^\epsilon_{xx} \qquad \text{with } \epsilon > 0$$

Multiply by $\eta'(q^\epsilon)$ to obtain:

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon \eta'(q^\epsilon) q^\epsilon_{xx}.$$

Entropy functions

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Multiply by $\eta'(q^\epsilon)$ to obtain:

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon \eta'(q^\epsilon) q_{xx}^\epsilon.$$

Manipulate further to get

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon\big(\eta'(q^\epsilon) q_x^\epsilon\big)_x - \epsilon \eta''(q^\epsilon)\,(q_x^\epsilon)^2.$$

Entropy functions

Smooth solution to viscous equation satisfies

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon\big(\eta'(q^\epsilon)q^\epsilon_x\big)_x - \epsilon\eta''(q^\epsilon)\,(q^\epsilon_x)^2.$$

Integrating over rectangle $[x_1, x_2] \times [t_1, t_2]$ gives

$$\begin{aligned}
\int_{x_1}^{x_2} \eta(q^\epsilon(x,t_2))\,dx &= \int_{x_1}^{x_2} \eta(q^\epsilon(x,t_1))\,dx \\
&- \left(\int_{t_1}^{t_2} \psi(q^\epsilon(x_2,t))\,dt - \int_{t_1}^{t_2} \psi(q^\epsilon(x_1,t))\,dt\right) \\
&+ \epsilon\int_{t_1}^{t_2} \big[\eta'(q^\epsilon(x_2,t))\,q^\epsilon_x(x_2,t) - \eta'(q^\epsilon(x_1,t))\,q^\epsilon_x(x_1,t)\big]\,dt \\
&- \epsilon\int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta''(q^\epsilon)\,(q^\epsilon_x)^2\,dx\,dt.
\end{aligned}$$

Let $\epsilon \to 0$ to get result:

Term on third line goes to 0,

Term of fourth line is always ≤ 0.

Entropy functions

Weak form of entropy condition:

$$\int_0^\infty \int_{-\infty}^\infty \left[\phi_t \eta(q) + \phi_x \psi(q)\right] dx\, dt + \int_{-\infty}^\infty \phi(x,0)\eta(q(x,0))\, dx \geq 0$$

for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R})$ with $\phi(x,t) \geq 0$ for all $x,\ t$.

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Informally we may write

$$\eta(q)_t + \psi(q)_x \leq 0.$$

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and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x,t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges (need stability).

Can also use FV version of entropy condition in weak form to show that limit must be correct weak solution.

And entropy stability can also be used to prove convergence.