Finite Volume Methods for Hyperbolic Problems

Dissipation, Dispersion, Modified Equations

- Upwind, Lax-Friedrichs
- Lax-Wendroff and Beam-Warming
- Numerical dissipation and dispersion
- Modified equations

Symmetric methods

Centered in space, forward in time:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n)$$

Flux differencing with $\mathcal{F}(Q_{i-1}, Q_i) = \frac{1}{2}(AQ_{i-1} + AQ_i)$ for $f(q) = Aq$.

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Lax-Friedrichs:

$$Q_i^{n+1} = \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n)$$

This is stable if $\left|\frac{\lambda^p \Delta t}{\Delta x}\right| \leq 1$ for all p.

Numerical dissipation

Lax-Friedrichs:

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This can be rewritten as

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

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&= Q_i^n - \Delta t A\left(\frac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x}\right) + \Delta t\left(\frac{\Delta x^2}{2\Delta t}\right)\left(\frac{Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n}{\Delta x^2}\right)
\end{aligned}$$

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The unstable method with the addition of artificial viscosity,

Approximates $q_t + Aq_x = \epsilon q_{xx}$ (modified equation)

with $\epsilon = \frac{\Delta x^2}{2\Delta t} = \mathcal{O}(\Delta x)$ if $\Delta t/\Delta x$ is fixed as $\Delta x \to 0$.

Modified Equations

The upwind method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} u(Q_i^n - Q_{i-1}^n).$$

gives a first-order accurate approximation to $q_t + uq_x = 0$.

But it gives a second-order approximation to

$$q_t + uq_x = \frac{u\Delta x}{2}\left(1 - \frac{u\Delta t}{\Delta x}\right) q_{xx}.$$

This is an advection-diffusion equation.

Indicates that the numerical solution will diffuse.

Note: coefficient of diffusive term is $O(\Delta x)$.

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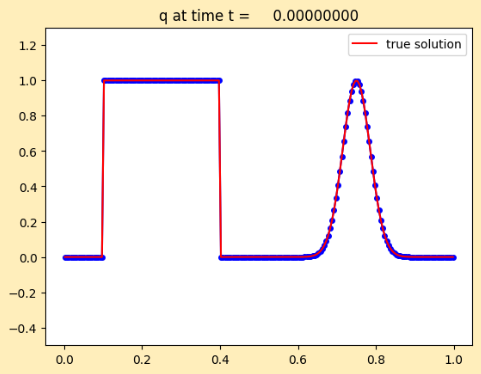
Note: No diffusion if $\frac{u\Delta t}{\Delta x} = 1 \quad (Q_i^{n+1} = Q_{i-1}^n$ exactly).

Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Initial data with 200 cells:



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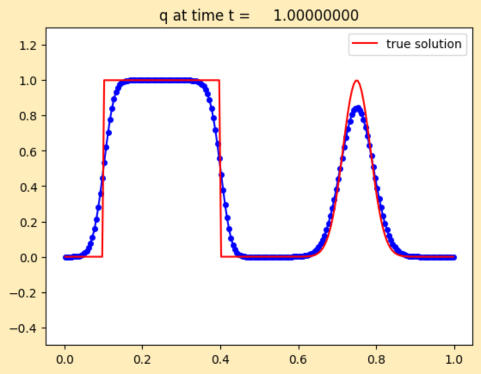
Upwind solution with 200 cells:



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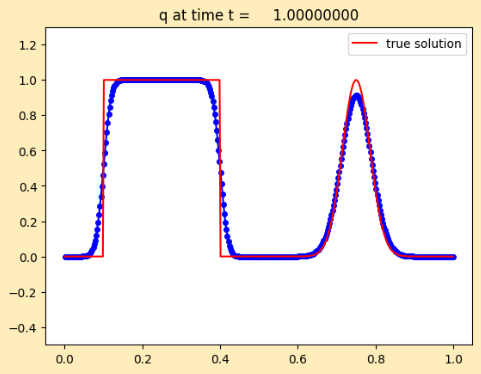
Upwind solution with 400 cells:



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Lax-Wendroff

Second-order accuracy?

Taylor series:

$$q(x, t + \Delta t) = q(x, t) + \Delta t q_t(x, t) + \frac{1}{2}\Delta t^2 q_{tt}(x, t) + \cdots$$

From $q_t = -Aq_x$ we find $q_{tt} = A^2 q_{xx}$.

$$q(x, t + \Delta t) = q(x, t) - \Delta t A q_x(x, t) + \frac{1}{2}\Delta t^2 A^2 q_{xx}(x, t) + \cdots$$

Replace q_x and q_{xx} by centered differences:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2 (Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

Modified Equation for Lax-Wendroff

The Lax-Wendroff method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2 (Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

gives a second-order accurate approximation to $q_t + uq_x = 0$.

But it gives a third-order approximation to

$$q_t + uq_x = -\frac{u\Delta x^2}{6}\left(1 - \left(\frac{u\Delta t}{\Delta x}\right)^2\right) q_{xxx}.$$

This has a dispersive term with $O(\Delta x^2)$ coefficient.

Indicates that the numerical solution will become oscillatory.

Dispersion relation

Consider a single Fourier mode:

$$q(x,0) = e^{i\xi x} \implies q(x,t) = e^{i(\xi x - \omega t)}$$

Determine $\omega(\xi)$ based on the PDE (dispersion relation)

$q_t = -i\omega q, \quad q_x = i\xi q,$

$q_t + uq_x = 0 \implies \omega(\xi) = u\xi, \qquad q(x,t) = e^{i\xi(x-ut)}$

(translates at speed u for all ξ)

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$q_t + uq_x = \epsilon q_{xx} \implies \qquad q(x,t) = e^{-\epsilon\xi^2 t} e^{i\xi(x-ut)}$ (decays)

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Consider a single Fourier mode:

$$q(x,0) = e^{i\xi x} \implies q(x,t) = e^{i(\xi x - \omega t)}$$

Determine $\omega(\xi)$ based on the PDE (dispersion relation)

$$q_t = -i\omega q, \quad q_x = i\xi q, \quad q_{xx} = -\xi^2 q, \quad q_{xxx} = -i\xi^3 q, \ldots$$

$$q_t + uq_x = 0 \implies \omega(\xi) = u\xi, \qquad q(x,t) = e^{i\xi(x-ut)}$$

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$$q_t + uq_x = \epsilon q_{xx} \implies \qquad q(x,t) = e^{-\epsilon\xi^2 t} e^{i\xi(x-ut)}$$ (decays)

$$q_t + uq_x = \beta q_{xxx} \implies \quad q(x,t) = e^{i\xi(x-(u+\beta\xi^2)t)}$$

(translates at speed $u + \beta\xi^2$ that depends on wave number!)

Advection tests

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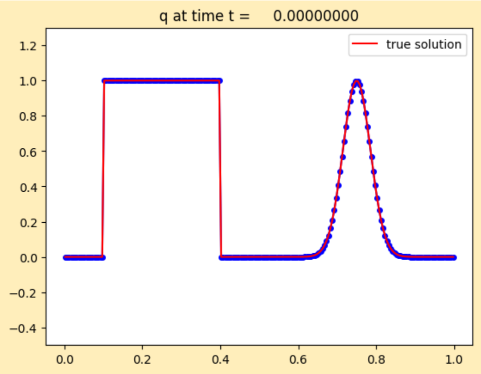
Initial data with 200 cells:



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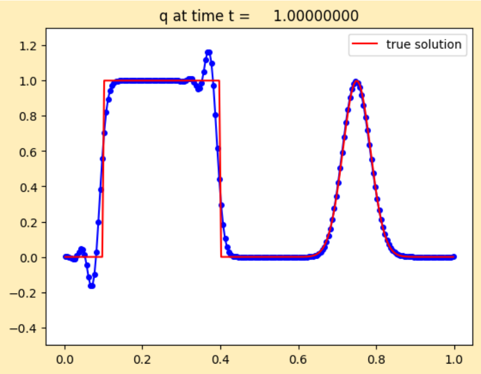
Lax-Wendroff solution with 200 cells:



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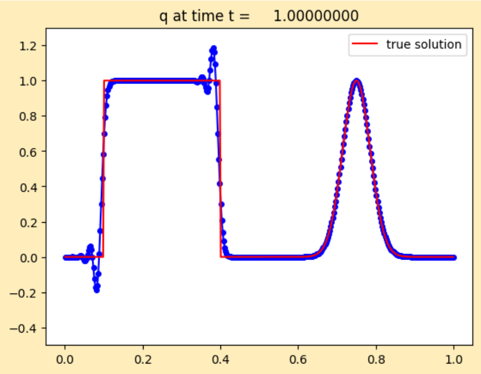
Lax-Wendroff solution with 400 cells:



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Beam-Warming method

Taylor series for second order accuracy:

$$q(x, t + \Delta t) = q(x, t) - \Delta t A q_x(x, t) + \frac{1}{2}\Delta t^2 A^2 q_{xx}(x, t) + \cdots$$

Replace q_x and q_{xx} by one-sided differences:

$$\begin{aligned} Q_i^{n+1} = Q_i^n &- \frac{\Delta t}{2\Delta x} A(3Q_i^n - 4Q_{i-1}^n + Q_{i-2}^n) \\ &+ \frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2(Q_i^n - 2Q_{i-1}^n + Q_{i-2}^n) \end{aligned}$$

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CFL condition: $0 \le \frac{\lambda^p \Delta t}{\Delta x} \le 2$ for all eigenvalues.

This is also the stability limit (von Neumann analysis).

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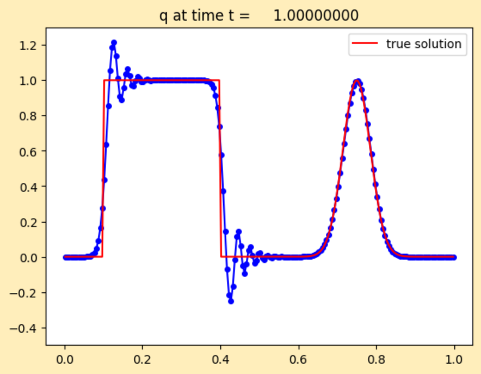

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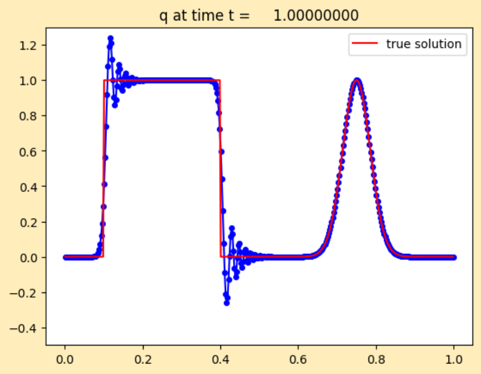

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