Finite Volume Methods for Hyperbolic Problems

Introduction to Finite Volume Methods

- Comparsion to finite differences
- Conservation form, importance for shocks
- Godunov's method, wave propagation view
- Upwind for advection
- REA Algorithm
- Godunov applied to acoustics

Finite difference method

Based on point-wise approximations:

$$Q_i^n \approx q(x_i, t_n), \quad \text{with } \; x_i = i\Delta x, \;\; t_n = n\Delta t.$$

Approximate derivatives by finite differences.

Ex: Upwind method for advection equation if $u > 0$:

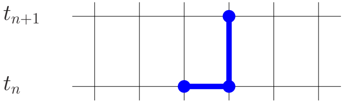
$$q_t + uq_x = 0$$

$$\frac{Q_i^{n+1} - Q_i^n}{\Delta t} + u\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right) = 0$$

or

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} u(Q_i^n - Q_{i-1}^n).$$

Stencil:



Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx \quad = \quad f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Finite volume method

$Q_i^n \approx \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$

Integral form: $$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

Finite volume method

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Integrate from t_n to t_{n+1} $\Longrightarrow$

$$\int q(x, t_{n+1})\, dx = \int q(x, t_n)\, dx + \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))\, dt$$

Finite volume method

$Q_i^n \approx \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)\,dx$

Integral form: $$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

Integrate from t_n to t_{n+1} $\implies$

$$\int q(x,t_{n+1})\,dx = \int q(x,t_n)\,dx + \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))\,dt$$

$$\begin{aligned}\frac{1}{\Delta x}\int q(x,t_{n+1})\,dx = {} & \frac{1}{\Delta x}\int q(x,t_n)\,dx \\ & - \frac{\Delta t}{\Delta x}\left(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t)) - f(q(x_{i-1/2},t))\,dt\right)\end{aligned}$$

Numerical method: $$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

Numerical flux: $F_{i-1/2}^n \approx \dfrac{1}{\Delta t}\displaystyle\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,dt.$

Upwind for advection as a finite volume method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

$$F_{i-1/2} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} uq(x_{i-1/2}, t)\, dt.$$

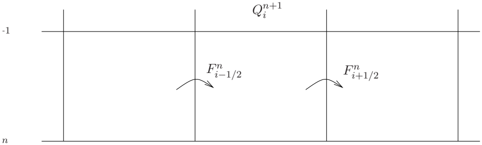
For $u > 0$:

$$F_{i-1/2}^n = uQ_{i-1}^n, \qquad F_{i+1/2}^n = uQ_i^n$$

so

$$\begin{aligned} Q_i^{n+1} &= Q_i^n + \frac{\Delta t(uQ_{i-1}^n - uQ_i^n)}{\Delta x} \\ &= Q_i^n - \frac{\Delta t u}{\Delta x}(Q_i^n - Q_{i-1}^n) \end{aligned}$$

Stencil:
(x-t plane)



Upwind method for advection

Flux: $f(q) = uq$

Numerical flux: $F^n_{i-1/2} \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}, t))\, dt.$

If $q(x, t_n)$ is piecewise constant in each cell, then

$$F^n_{i-1/2} = \begin{cases} uQ^n_{i-1} & \text{if } u > 0, \\ uQ^n_i & \text{if } u < 0. \end{cases}$$

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$$F^n_{i-1/2} = \begin{cases} uQ^n_{i-1} & \text{if } u > 0, \\ uQ^n_i & \text{if } u < 0. \end{cases}$$

This gives the upwind method:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) \qquad \text{if } u > 0$$

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_{i+1}^n - Q_i^n) \qquad \text{if } u < 0$$

Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \frac{\Delta t}{\Delta x}(F_{+\infty} - F_{-\infty}).$$

Note: an isolated shock must travel at the right speed!

Nonlinear scalar conservation laws

Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = 0$.

Quasilinear form: $u_t + uu_x = 0$.

These are equivalent for smooth solutions, not for shocks!

Nonlinear scalar conservation laws

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These are equivalent for smooth solutions, not for shocks!

Upwind methods for $u > 0$:

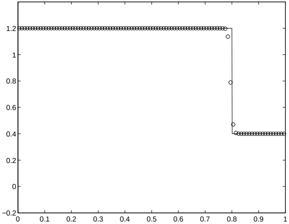
Conservative: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}((U_i^n)^2 - (U_{i-1}^n)^2)\right)$

Quasilinear: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n)$.

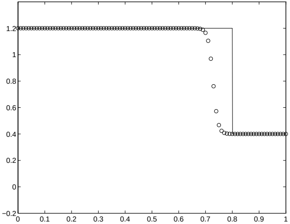
Ok for smooth solutions, not for shocks!

Importance of conservation form

Solution to Burgers' equation using conservative upwind:



Solution to Burgers' equation using quasilinear upwind:



Weak solutions depend on the conservation law

The conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0 \qquad \text{i.e.} \qquad q = u^2,\ f(q) = \frac{2}{3}q^{3/2}$$

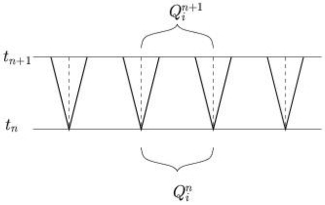
both have the same quasilinear form

$$u_t + uu_x = 0$$

but have different weak solutions,

different shock speeds!

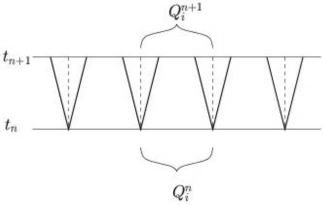
Godunov's Method for $q_t + f(q)_x = 0$



1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}^p_{i-1/2}$ and speeds $s^p_{i-1/2}$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

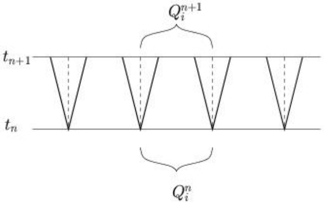
Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},

Godunov's Method for $q_t + f(q)_x = 0$

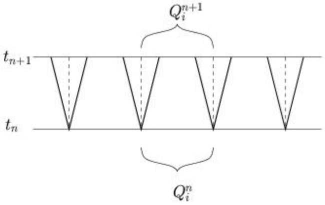


Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

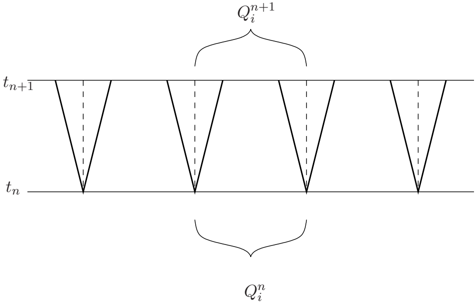
where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

Godunov's method with flux differencing

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\implies$ Riemann problems.



$$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\Downarrow}(Q_{i-1}, Q_i) \quad \text{for } t > t_n.$$

$$F_{i-1/2}^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n))\, dt = f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n)).$$

Upwind method for advection

Flux: $f(q) = uq$

Numerical flux: $$F^n_{i-1/2} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,dt.$$

If $q(x,t_n)$ is piecewise constant in each cell, then

$$F^n_{i-1/2} = \begin{cases} uQ^n_{i-1} & \text{if } u > 0, \\ uQ^n_i & \text{if } u < 0. \end{cases}$$

Upwind method for advection

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If $q(x,t_n)$ is piecewise constant in each cell, then

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This gives the upwind method:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) \qquad \text{if } u > 0$$

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_{i+1}^n - Q_i^n) \qquad \text{if } u < 0$$

First-order REA Algorithm

1. Reconstruct a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

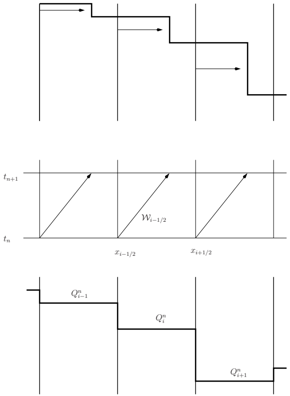
Godunov's method for advection

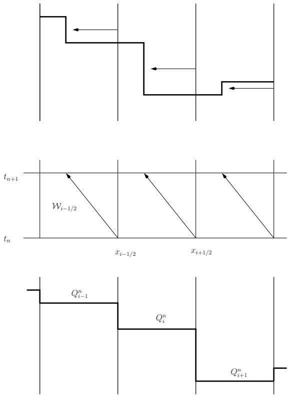
Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\implies$ Riemann problems.

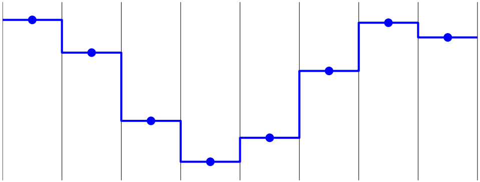
$u > 0$ $\qquad$ $u < 0$



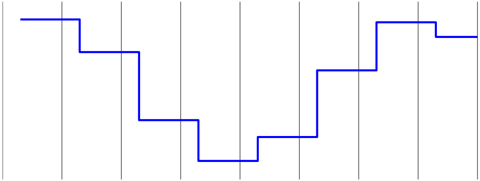


First-order REA Algorithm

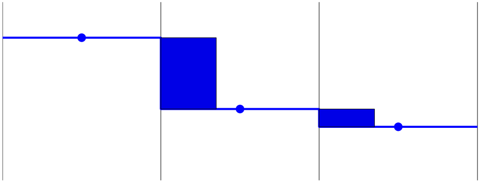
Cell averages and piecewise constant reconstruction:



After evolution:



Cell update



The cell average is modified by

$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x}$$

So we obtain the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n).$$

Wave propagation form of cell update

The cell average is modified by

$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x} = -\frac{\Delta t}{\Delta x} s\mathcal{W}_{i-1/2}$$

where $\mathcal{W}_{i-1/2} = (Q_i^n - Q_{i-1}^n)$ is the wave strength and $s = u$ is the wave speed.

The general upwind method for $u < 0$ or $u > 0$:

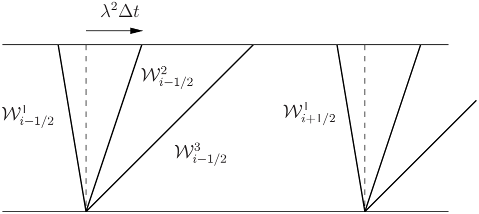
$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{\Delta t}{\Delta x}\left[u^+(Q_i^n - Q_{i-1}^n) + u^-(Q_{i+1}^n - Q_i^n)\right] \\ &= \frac{\Delta t}{\Delta x}\left[s^+\mathcal{W}_{i-1/2} + s^-\mathcal{W}_{i-1/2}\right] \end{aligned}$$

where $u^+ = \max(u, 0), \;\; u^- = \min(u, 0)$.
This is the wave propagation form of upwind.

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of

waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

Godunov (upwind) for a linear system

$q_t + Aq_x = 0$ where $A = R\Lambda R^{-1}$. Define the matrices

$$\Lambda^+ = \begin{bmatrix} (\lambda^1)^+ & & & \\ & (\lambda^2)^+ & & \\ & & \ddots & \\ & & & (\lambda^m)^+ \end{bmatrix}, \qquad \Lambda^- = \begin{bmatrix} (\lambda^1)^- & & & \\ & (\lambda^2)^- & & \\ & & \ddots & \\ & & & (\lambda^m)^- \end{bmatrix}.$$

and

$$A^+ = R\Lambda^+ R^{-1}, \qquad \text{and} \qquad A^- = R\Lambda^- R^{-1}.$$

Note:

$$A^+ + A^- = R(\Lambda^+ + \Lambda^-)R^{-1} = R\Lambda R^{-1} = A.$$

Then Godunov's method becomes

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+(Q_i - Q_{i-1}) + A^-(Q_{i+1} - Q_i)\right].$$

Matrix splitting for upwind method

For $q_t + Aq_x = 0$, the upwind method (Godunov) is:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n + \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\alpha_{i-1/2}^p r^p + \sum_{p=1}^{m}(\lambda^p)^-\alpha_{i+1/2}^p r^p\right] \\ &= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right] \\ &= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)\right] \end{aligned}$$

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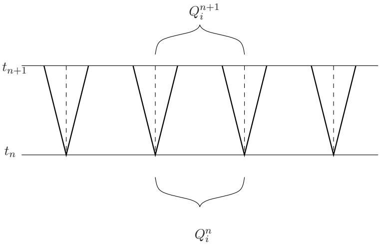
Natural generalization of upwind to a system.

If all eigenvalues are positive, then $A^+ = A$ and $A^- = 0$,

If all eigenvalues are negative, then $A^+ = 0$ and $A^- = A$.

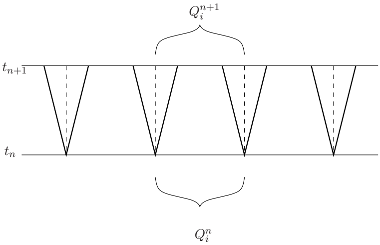
Godunov (upwind) on acoustics



Data at time t_n : $\tilde{q}^n(x, t_n) = Q_i^n$ for $x_{i-1/2} < x < x_{i+1/2}$
Solving Riemann problems for small Δt gives solution:

$$\tilde{q}^n(x, t_{n+1}) = \begin{cases} Q^*_{i-1/2} & \text{if } x_{i-1/2} - c\Delta t < x < x_{i-1/2} + c\Delta t, \\ Q_i^n & \text{if } x_{i-1/2} + c\Delta t < x < x_{i+1/2} - c\Delta t, \\ Q^*_{i+1/2} & \text{if } x_{i+1/2} - c\Delta t < x < x_{i+1/2} + c\Delta t, \end{cases}$$

Godunov (upwind) on acoustics



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So computing cell average gives:

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right].$$

Godunov (upwind) on acoustics

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Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2} = \alpha^1_{i-1/2} r^1 + \alpha^2_{i-1/2} r^2,$$

$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}^1_{i+1/2} + \mathcal{W}^2_{i+1/2} = \alpha^1_{i+1/2} r^1 + \alpha^2_{i+1/2} r^2,$$

Godunov (upwind) on acoustics

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right].$$

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2} = \alpha^1_{i-1/2} r^1 + \alpha^2_{i-1/2} r^2,$$
$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}^1_{i+1/2} + \mathcal{W}^2_{i+1/2} = \alpha^1_{i+1/2} r^1 + \alpha^2_{i+1/2} r^2,$$

The intermediate states are:

$$Q^*_{i-1/2} = Q_i^n - \mathcal{W}^2_{i-1/2}, \qquad Q^*_{i+1/2} = Q_i^n + \mathcal{W}^1_{i+1/2},$$

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$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right].$$

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2} = \alpha^1_{i-1/2} r^1 + \alpha^2_{i-1/2} r^2,$$
$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}^1_{i+1/2} + \mathcal{W}^2_{i+1/2} = \alpha^1_{i+1/2} r^1 + \alpha^2_{i+1/2} r^2,$$

The intermediate states are:

$$Q^*_{i-1/2} = Q_i^n - \mathcal{W}^2_{i-1/2}, \qquad Q^*_{i+1/2} = Q_i^n + \mathcal{W}^1_{i+1/2},$$

So,

$$\begin{aligned} Q_i^{n+1} &= \frac{1}{\Delta x}\left[c\Delta t(Q_i^n - \mathcal{W}^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t(Q_i^n + \mathcal{W}^1_{i+1/2})\right] \\ &= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2} + \frac{c\Delta t}{\Delta x}\mathcal{W}^1_{i+1/2}. \end{aligned}$$

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$$
\begin{aligned}
Q_i^{n+1} &= \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right] \\
&= \frac{1}{\Delta x}\left[c\Delta t (Q_i^n - \mathcal{W}^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t (Q_i^n + \mathcal{W}^1_{i+1/2})\right] \\
&= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2} + \frac{c\Delta t}{\Delta x}\mathcal{W}^1_{i+1/2} \\
&= Q_i^n - \frac{\Delta t}{\Delta x}(c\mathcal{W}^2_{i-1/2} + (-c)\mathcal{W}^1_{i+1/2}).
\end{aligned}
$$

Godunov (upwind) on acoustics

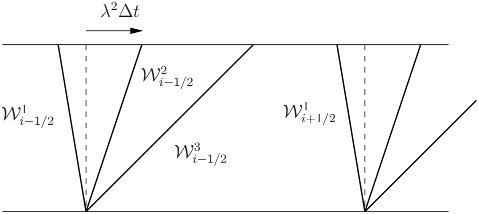
$$
\begin{aligned}
Q_i^{n+1} &= \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right] \\
&= \frac{1}{\Delta x}\left[c\Delta t(Q_i^n - \mathcal{W}^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t(Q_i^n + \mathcal{W}^1_{i+1/2})\right] \\
&= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2} + \frac{c\Delta t}{\Delta x}\mathcal{W}^1_{i+1/2} \\
&= Q_i^n - \frac{\Delta t}{\Delta x}(c\mathcal{W}^2_{i-1/2} + (-c)\mathcal{W}^1_{i+1/2}).
\end{aligned}
$$

General form for linear system with m equations:

$$
\begin{aligned}
Q_i^{n+1} &= Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p:\lambda^p>0}\lambda^p\mathcal{W}^p_{i-1/2} + \sum_{p:\lambda^p<0}\lambda^p\mathcal{W}^p_{i+1/2}\right] \\
&= Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{m=1}^{p}(\lambda^p)^+\mathcal{W}^p_{i-1/2} + \sum_{m=1}^{p}(\lambda^p)^-\mathcal{W}^p_{i+1/2}\right]
\end{aligned}
$$

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$