Finite Volume Methods for Hyperbolic Problems

Linear Systems – Nonhyperbolic Cases

- Acoustics equations if $K_0 < 0$ (eigenvalues complex)
- Acoustics equations if $K_0 = 0$ (not diagonalizable)
- Coupled advection equations

Linear acoustics

Example: Linear acoustics in a 1d gas tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{aligned} p(x,t) &= \text{pressure perturbation} \\ u(x,t) &= \text{velocity} \end{aligned}$$

Equations:

$$\begin{aligned} p_t + K_0 u_x &= 0 \qquad \text{Change in pressure due to compression} \\ \rho_0 u_t + p_x &= 0 \qquad \text{Newton's second law, } F = ma \end{aligned}$$

where $K_0 =$ bulk modulus, and $\rho_0 =$ unperturbed density.

Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Acoustics equations when hyperbolicity fails

Eigenvalues are $\pm\sqrt{K_0/\rho}$ (wave speeds),
real and distinct provided $K_0 > 0$ and $\rho_0 > 0$.

Now suppose $K_0 < 0$. Then eigenvalues pure imaginary.

Recall $K_0 = \rho_0 P'(\rho_0)$ from linearization.

Physically we expect pressure to increase as density increases.

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Physically we expect pressure to increase as density increases.

Otherwise, mass flowing in leads to decreased pressure and hence greater mass flow, with mass growing exponentially without bound.

Second-order PDE form of acoustics

$$\begin{aligned} p_t + K_0 u_x = 0 \quad &\implies p_{tt} = -K_0 u_{xt} \\ u_t + (1/\rho_0) u_x = 0 \quad &\implies u_{tx} = -(1/\rho_0) p_{xx} \end{aligned}$$

Combining gives

$$p_{tt} = c_0^2 p_{xx}$$

with $c_0^2 = K_0/\rho_0$. This is the wave equation provided $c_0^2 > 0$.

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To solve for $x_1 \le x \le x_2$ and $t_0 \le t \le T$, the elliptic equation requires BCs on all four sides, including at $t = T$.

The initial-boundary value problem is ill-posed.

Acoustics equations when hyperbolicity fails

Eigenvalues are $\pm\sqrt{K_0/\rho_0}$ (wave speeds).

Now suppose $K_0 = 0$. Then eigenvalues are $\lambda^1 = \lambda^2 = 0$.
Wave speeds are 0, not necessarily a problem.

But the matrix is a Jordan block, not diagonalizable:

$$A = \begin{bmatrix} 0 & 0 \\ 1/\rho_0 & 0 \end{bmatrix}.$$

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Equations become:

$$\begin{aligned} p_t &= 0, \\ u_t &= -(1/\rho_0)p_x. \end{aligned}$$

$p(x,t) = \overset{\circ}{p}(x)$ for all time
u_t can grow arbitrarily quickly depending on $\overset{\circ}{p}_x$. Ill-posed.

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In particular, Riemann problem can have infinite p_x at origin.

Acoustics equations in limit $K_0 = K \to 0$

$$A = \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix}, \quad \text{Eigenvalues: } \lambda = \pm\sqrt{K/\rho} \to 0.$$

Impedance $Z = \sqrt{K\rho} \to 0$.

$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2} \begin{bmatrix} (p_l + p_r) - Z(u_r - u_l) \\ (u_l + u_r) - (p_r - p_l)/Z \end{bmatrix}.$$

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So if $p_r \neq p_l$, then $u_m \to \infty$ as $K \to 0$

Another non-diagonalizable example (Sec. 16.3.1)

$$
\begin{aligned}
q_t^1 + uq_x^1 + \beta q_x^2 &= 0, \\
q_t^2 \qquad\qquad + vq_x^2 &= 0,
\end{aligned}
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has

$$A = \begin{bmatrix} u & \beta \\ 0 & v \end{bmatrix}.$$

Eigenvalues and eigenvectors (if $v \leq u$ and $\beta \neq 0$):

$$\lambda^1 = v, \qquad \lambda^2 = u,$$

$$r^1 = \begin{bmatrix} \beta \\ v - u \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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As $u \to v$ the eigenvector r^1 becomes colinear with r^2 and the eigenvector matrix R becomes singular (unless $\beta = 0$).