Finite Volume Methods for Hyperbolic Problems

Linear Systems – Riemann Problems

- Riemann problems
- Riemann problem for advection
- Riemann problem for acoustics
- Phase plane

The Riemann problem

The Riemann problem consists of the hyperbolic equation under study together with initial data of the form

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

Piecewise constant with a single jump discontinuity from q_l to q_r.

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The Riemann problem is fundamental to understanding

- The mathematical theory of hyperbolic problems,
- Godunov-type finite volume methods

Why? Even for nonlinear systems of conservation laws, the Riemann problem can often be solved for general q_l and q_r, and consists of a set of waves propagating at constant speeds.

The Riemann problem for advection

The Riemann problem for the advection equation $q_t + uq_x = 0$ with

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

has solution

$$q(x,t) = q(x-ut,0) = \begin{cases} q_l & \text{if } x < ut \\ q_r & \text{if } x \geq ut \end{cases}$$

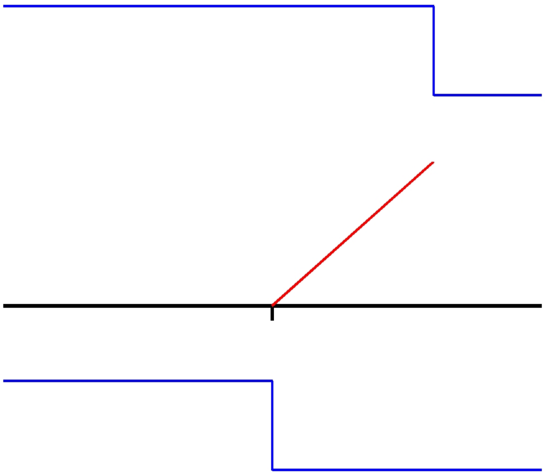
consisting of a single wave of strength $\mathcal{W}^1 = q_r - q_l$ propagating with speed $s^1 = u$.

Riemann solution for advection

$q(x, T)$

x–t plane

$q(x, 0)$



Discontinuous solutions

Note: The Riemann solution is not a classical solution of the PDE $q_t + uq_x = 0$, since q_t and q_x blow up at the discontinuity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t)$$

Integrate in time from t_1 to t_2 to obtain

$$\begin{aligned}\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx \\ = \int_{t_1}^{t_2} uq(x_1,t)\,dt - \int_{t_1}^{t_2} uq(x_2,t)\,dt.\end{aligned}$$

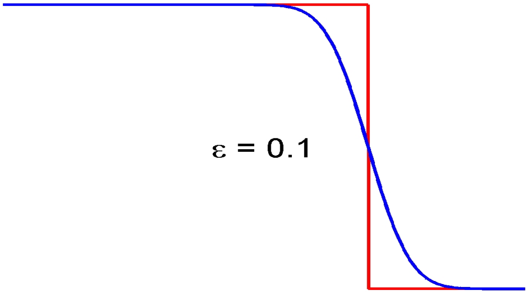
The Riemann solution satisfies the given initial conditions and this integral form for all $x_2 > x_1$ and $t_2 > t_1 \geq 0$.

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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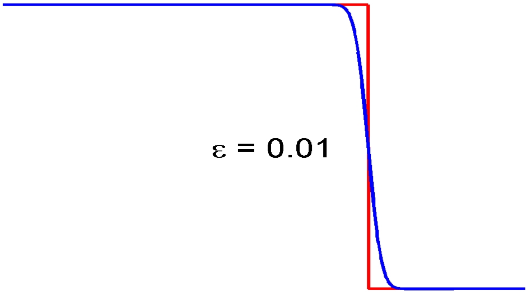

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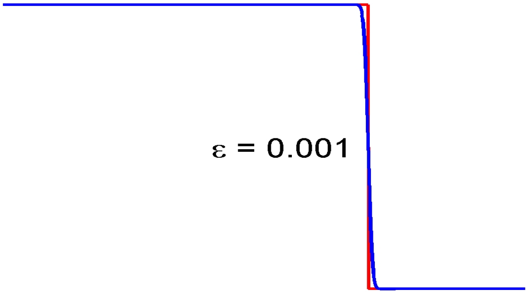

Riemann Problems and Jupyter Solutions

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Eigenvectors for acoustics

$$A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $Ar^p = \lambda^p r^p$, where

$$\lambda^1 = -c_0, \qquad \lambda^2 = +c_0.$$

with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

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Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} =$ impedance.

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

Consider a pure 1-wave (simple wave), at speed $\lambda^1 = -c_0$,
If $\overset{\circ}{q}(x) = \bar{q} + \overset{\circ}{w}^1(x) r^1$ then

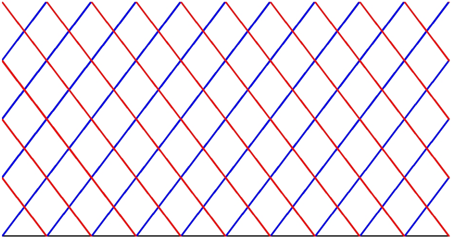
$$q(x,t) = \bar{q} + \overset{\circ}{w}^1(x - \lambda^1 t) r^1$$

Variation of q, as measured by q_x or $\Delta q = q(x + \Delta x) - q(x)$ is proportional to eigenvector r^1, e.g.

$$q_x(x,t) = \overset{\circ}{w}^1_x(x - \lambda^1 t) r^1$$

Linear acoustics — characteristics

$$
\begin{aligned}
q(x,t) &= w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2 \\
&= \frac{-\overset{\circ}{p}(x+ct)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x-ct)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.
\end{aligned}
$$



For IBVP on $a < x < b$, must specify one incoming boundary condition at each side: $w^2(a,t)$ and $w^1(b,t)$

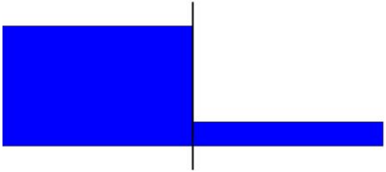
Riemann Problem

Special initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphram ($u_l = u_r = 0$)

Pressure:



Acoustic waves propagate with speeds $\pm c$.

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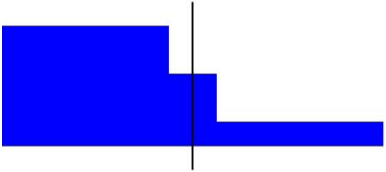

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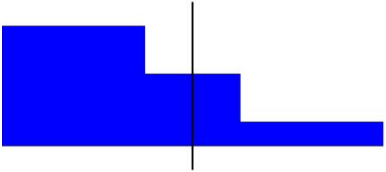

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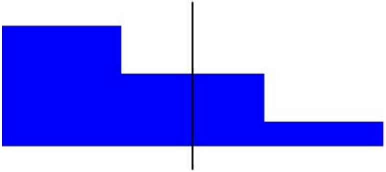

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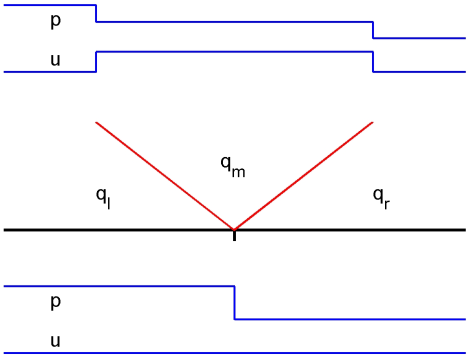
Riemann Problem for acoustics

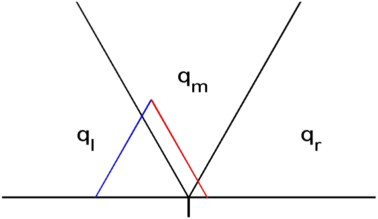
Waves propagating in x–t space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

Riemann Problem for acoustics

In x–t plane:



$$q(x,t) = w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2$$

Decompose q_l and q_r into eigenvectors:

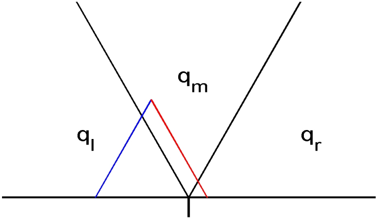
$$q_l = w_l^1 r^1 + w_l^2 r^2$$
$$q_r = w_r^1 r^1 + w_r^2 r^2$$

Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

Riemann Problem for acoustics

In x–t plane:



Decompose $q_r - q_l$ into eigenvectors: Solve $R\alpha = \Delta q$

$$\begin{aligned} q_r - q_l &= (w_r^1 - w_r^1)r^1 + (w_r^2 - w_r^2)r^2 \\ &= \alpha^1 r^1 + \alpha^2 r^2 \quad = \mathcal{W}^1 + \mathcal{W}^2. \end{aligned}$$

Then

$$\begin{aligned} q_m &= w_r^1 r^1 + w_l^2 r^2 \\ &= q_l + \alpha^1 r^1 \qquad = q_r - \alpha^2 r^2. \end{aligned}$$

Riemann solution for acoustics

$$r^1 = \begin{bmatrix} -\rho c \\ 1 \end{bmatrix} = \begin{bmatrix} -Z \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho c \\ 1 \end{bmatrix} = \begin{bmatrix} Z \\ 1 \end{bmatrix}.$$

Solving $R\alpha = \Delta q$ gives:

$$\alpha^1 = \frac{-\Delta p + Z\Delta u}{2Z}, \qquad \alpha^2 = \frac{\Delta p + Z\Delta u}{2Z},$$

so

$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2}\begin{bmatrix} (p_l + p_r) - Z(u_r - u_l) \\ (u_l + u_r) - (p_r - p_l)/Z \end{bmatrix}.$$

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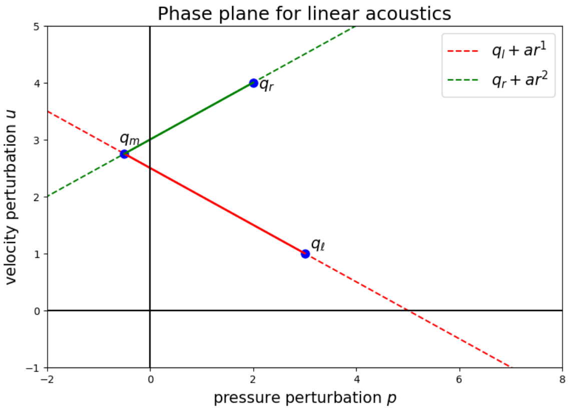
Ex: shock tube with $u_l = u_r = 0$:

$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2}\begin{bmatrix} (p_l + p_r) \\ -(p_r - p_l)/Z \end{bmatrix}.$$

Phase plane solution to Riemann problem



q_ℓ and q_m are connected by a multiple of r^1
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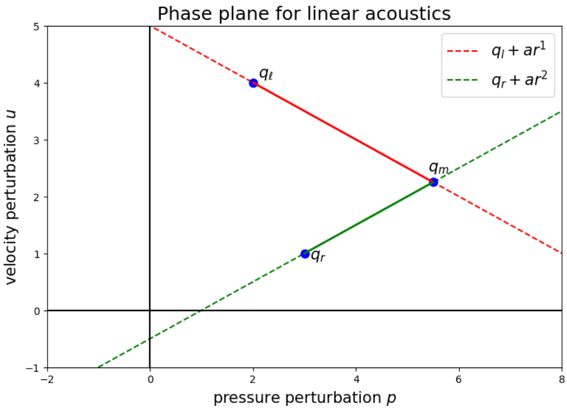

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Note that swapping q_ℓ and q_r changes the solution!

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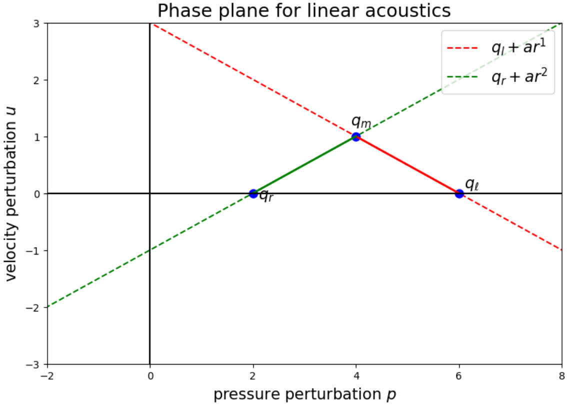

"Shock tube" solution with $u_\ell = u_r = 0$.

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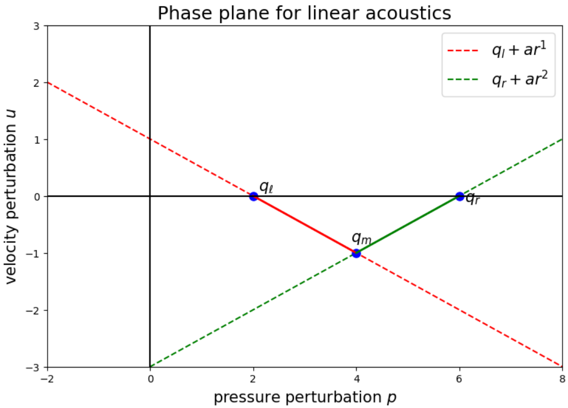

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Riemann solution for a linear system

Linear hyperbolic system: $q_t + Aq_x = 0$ with $A = R\Lambda R^{-1}$.
General Riemann problem data $q_l,\ q_r \in \mathbb{R}^m$.

Decompose jump in q into eigenvectors:

$$q_r - q_l = \sum_{p=1}^{m} \alpha^p r^p$$

Note: the vector α of eigen-coefficients is

$$\alpha = R^{-1}(q_r - q_l) = R^{-1}q_r - R^{-1}q_l = w_r - w_l.$$

Riemann solution consists of m waves $\mathcal{W}^p \in \mathbb{R}^m$:

$$\mathcal{W}^p = \alpha^p r^p, \qquad \text{propagating with speed } \ s^p = \lambda^p.$$

Phase space

For a system of m equations, phase space is m-dimensional.

Solving the Riemann problem finds a path from q_ℓ to q_r that generally has m segments, each in the direction of an eigenvector (for a linear system; curves more generally).

If $\lambda^1 \le \lambda^2 \le \cdots \le \lambda^m$, then first segment from q_ℓ to $q_\ell + \alpha^1 r^1$, next segment goes to $q_\ell + \alpha^1 r^1 + \alpha^2 r^2$, etc.

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Unique such path provided eigenvectors are linearly independent. $q_\ell + \alpha^1 r^1 + \alpha^2 r^2 + \cdots + \alpha^m r^m = q_r$.

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Visualization is most useful when $m = 2$ (phase plane).

But sometimes illuminating to project phase space onto a two-dimensional plane.