Finite Volume Methods for Hyperbolic Problems

Linear Hyperbolic Systems

- General form, coefficient matrix, hyperbolicity
- Scalar advection equation
- Linear acoustics equations
- Eigen decomposition
- Characteristics and general solution
- Boundary conditions

Linear hyperbolic systems

Linear system of m equations: $q(x,t) \in \mathbb{R}^m$ for each (x,t) and

$$q_t + Aq_x = 0, \qquad -\infty < x, \infty, \;\; t \geq 0.$$

A is $m \times m$ matrix (constant for now, independent of x, t)

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Eigenvalues are wave speeds.

Eigenvectors used to split arbibrary data into waves.
So matrix of eigenvectors must be nonsingular.

Advection equation as a linear system

$$q_t + uq_x = 0$$

with u a constant (real) velocity. (1×1 diagonalizable, $\lambda^1 = u$)

Initial condition:

$$q(x,0) = \overset{\circ}{q}(x), \quad -\infty < x < \infty.$$

The solution to this Cauchy problem is:

$$q(x,t) = \overset{\circ}{q}(x - ut)$$

It is constant along each characteristic curve

$$X(t) = x_0 + ut$$

Characteristics for advection

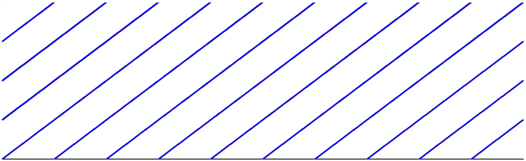
$q(x,t) = \overset{\circ}{q}(x-ut) \implies q$ is constant along lines

$$X(t) = x_0 + ut, \quad t \geq 0.$$

Can also see that q is constant along $X(t)$ from:

$$\begin{aligned} \frac{d}{dt} q(X(t),t) &= q_x(X(t),t)X'(t) + q_t(X(t),t) \\ &= q_x(X(t),t)u + q_t(X(t),t) \\ &= 0. \end{aligned}$$

In x–t plane:



Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Suppose hyperbolic:

- Real eigenvalues $\lambda^1 \le \lambda^2 \le \cdots \le \lambda^m$,
- Linearly independent eigenvectors $r^1,\ r^2,\ \ldots,\ r^m$.

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Let $R = [r^1|r^2|\cdots|r^m] \quad m \times m$ matrix of eigenvectors.

Then $Ar^p = \lambda^p r^p$ means that $AR = R\Lambda$ where

$$\Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix} \equiv \text{diag}(\lambda^1, \lambda^2, \ldots, \lambda^m).$$

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$AR = R\Lambda \implies A = R\Lambda R^{-1}$ and $R^{-1}AR = \Lambda$.
Similarity transformation with R diagonalizes A.

Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Multiply system by R^{-1}:

$$R^{-1}q_t(x,t) + R^{-1}Aq_x(x,t) = 0.$$

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Use $R^{-1}AR = \Lambda$ and define $w(x,t) = R^{-1}q(x,t)$:

$$w_t(x,t) + \Lambda w_x(x,t) = 0.$$

Since R is constant!

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$$w_t(x,t) + \Lambda w_x(x,t) = 0.$$ Since R is constant!

This decouples to m independent scalar advection equations:

$$w_t^p(x,t) + \lambda^p w_x^p(x,t) = 0. \qquad p = 1,\ 2,\ \ldots,\ m.$$

Solution to Cauchy problem

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The solution to the decoupled equation $w_t^p + \lambda^p w_x^p = 0$ is

$$w^p(x,t) = w^p(x - \lambda^p t, 0) = \overset{\circ}{w}{}^p(x - \lambda^p t).$$

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We can rewrite this as

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t)\, r^p = \sum_{p=1}^{m} \overset{\circ}{w}^p(x - \lambda^p t)\, r^p$$

Linear acoustics

Example: Linear acoustics in a 1d gas tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity} \end{array}$$

Equations:

$$\begin{aligned} p_t + K_0 u_x &= 0 \qquad && \text{Change in pressure due to compression} \\ \rho_0 u_t + p_x &= 0 && \text{Newton's second law, } F = ma \end{aligned}$$

where $K_0 =$ bulk modulus, and $\rho_0 =$ unperturbed density.

Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Eigenvectors for acoustics

$$A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $Ar^p = \lambda^p r^p$, where

$$\lambda^1 = -c_0, \qquad \lambda^2 = +c_0.$$

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Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} =$ impedance.

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

Consider a pure 1-wave (simple wave), at speed $\lambda^1 = -c_0$,
If $\overset{\circ}{q}(x) = \bar{q} + \overset{\circ}{w}^1(x) r^1$ then

$$q(x,t) = \bar{q} + \overset{\circ}{w}^1(x - \lambda^1 t) r^1$$

Variation of q, as measured by q_x or $\Delta q = q(x + \Delta x) - q(x)$ is proportional to eigenvector r^1, e.g.

$$q_x(x,t) = \overset{\circ}{w}^1_x(x - \lambda^1 t) r^1$$

Physical meaning of eigenvectors

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In a simple 1-wave (propagating at speed $\lambda^1 = -c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is $-Z_0$ times the velocity variation.

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Similarly, in a simple 2-wave ($\lambda^2 = c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is Z_0 times the velocity variation.

Acoustic waves

$$
\begin{aligned}
q(x,0) = \begin{bmatrix} p(x) \\ 0 \end{bmatrix} \quad &= \quad -\frac{p(x)}{2Z}\begin{bmatrix} -Z \\ 1 \end{bmatrix} \quad + \quad \frac{p(x)}{2Z}\begin{bmatrix} Z \\ 1 \end{bmatrix} \\
&= \quad w^1(x,0)r^1 \quad + \quad w^2(x,0)r^2 \\
&= \begin{bmatrix} p(x)/2 \\ -p(x)/(2Z) \end{bmatrix} + \begin{bmatrix} p(x)/2 \\ p(x)/(2Z) \end{bmatrix}.
\end{aligned}
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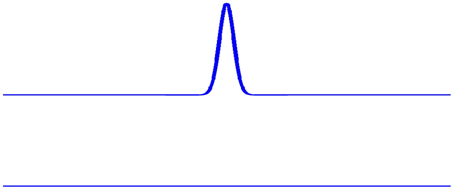

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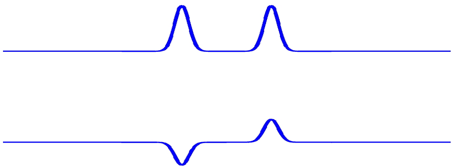

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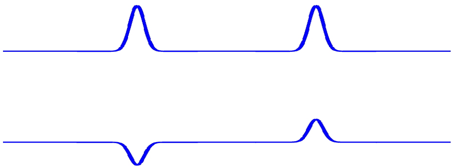

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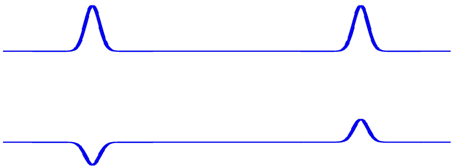

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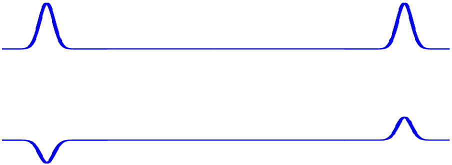

Solution by tracing back on characteristics

The general solution for acoustics:

$$\begin{aligned} q(x,t) &= w^1(x-\lambda^1 t,0)r^1 + w^2(x-\lambda^2 t,0)r^2 \\ &= w^1(x+c_0 t,0)r^1 + w^2(x-c_0 t,0)r^2 \end{aligned}$$

Recall that $w(x,0) = R^{-1}q(x,0)$, i.e.

$$w^1(x,0) = \ell^1 q(x,0), \qquad w^2(x,0) = \ell^2 q(x,0)$$

where ℓ^1 and ℓ^2 are rows of R^{-1}.

$$R^{-1} = \left[\begin{array}{c} \ell^1 \\ \ell^2 \end{array} \right]$$

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Note: ℓ^1 and ℓ^2 are left-eigenvectors of A:

$$\ell^p A = \lambda^p \ell^p \qquad \text{since } R^{-1}A = \Lambda R^{-1}.$$

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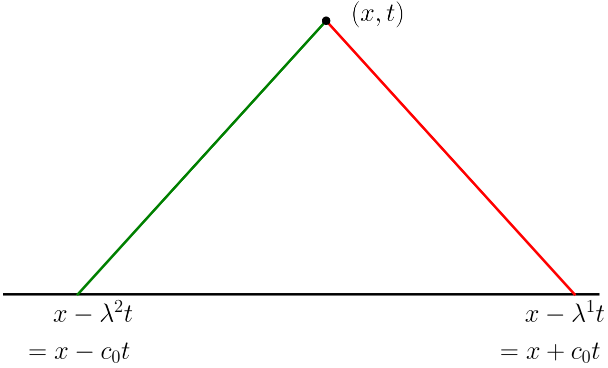

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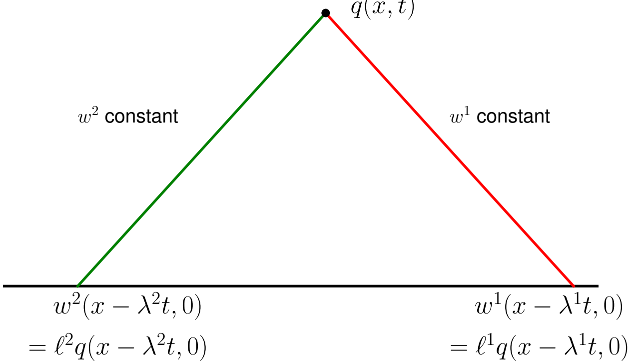

Linear acoustics

Example: Linear acoustics in a 1d gas tube,
linearized about $p = p_0,\ u = u_0$

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity perturbation} \end{array}$$

Equations include advective transport at speed u_0:

$$\begin{aligned} p_t + u_0 p_x + K_0 u_x &= 0 \qquad \text{Change in pressure due to compressio} \\ \rho_0 u_t + p_x + u_0 u_x &= 0 \qquad \text{Newton's second law, } F = ma \end{aligned}$$

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Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

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Eigenvectors:

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with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

Eigenvectors for acoustics

$$A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix} \qquad = u_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}$$

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Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} =$ impedance.

Initial–boundary value problem (IBVP) for advection

Advection equation on finite 1D domain:

$$q_t + uq_x = 0 \qquad a < x < b, \;\; t \geq 0,$$

with initial data

$$q(x,0) = \eta(x) \qquad a < x < b.$$

and boundary data at the inflow boundary:

If $u > 0$, need data at $x = a$:

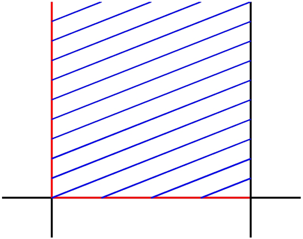
$$q(a,t) = g(t), \qquad t \geq 0,$$

If $u < 0$, need data at $x = b$:

$$q(b,t) = g(t), \qquad t \geq 0,$$

Characteristics for IBVP

In x–t plane for the case $u > 0$:



Solution:

$$q(x,t) = \begin{cases} \eta(x-ut) & \text{if } a \le x - ut \le b, \\ g((x-a)/u) & \text{otherwise} \end{cases}.$$

Periodic boundary conditions

$$q(a,t) = q(b,t), \qquad t \geq 0.$$

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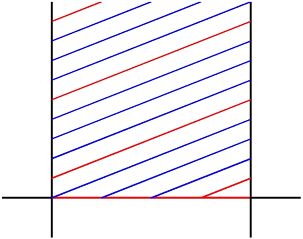


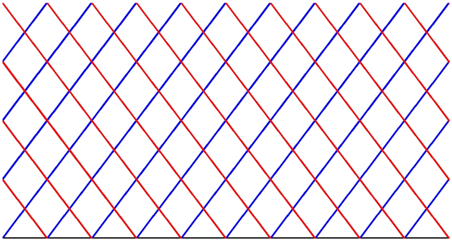
Solution:

$$q(x,t) = \eta(X_0(x,t)),$$

where $X_0(x,t) = a + \text{mod}(x - ut - a,\ b - a)$.

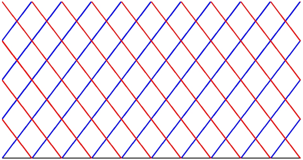
Linear acoustics — characteristics

$$
\begin{aligned}
q(x,t) &= w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2 \\
&= \frac{-\overset{\circ}{p}(x+ct)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x-ct)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.
\end{aligned}
$$



For IBVP on $a < x < b$, must specify one incoming boundary condition at each side: $w^2(a,t)$ and $w^1(b,t)$

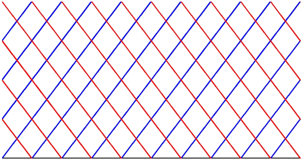
Acoustics boundary conditions



Outflow (non-reflecting, absorbing) boundary conditions:

$$w^2(a,t) = 0, \qquad w^1(b,t) = 0.$$

Acoustics boundary conditions



Outflow (non-reflecting, absorbing) boundary conditions:

$$w^2(a,t) = 0, \qquad w^1(b,t) = 0.$$

Periodic boundary conditions:

$$w^2(a,t) = w^2(b,t), \qquad w^1(b,t) = w^1(a,t),$$

or simply

$$q(a,t) = q(b,t).$$

Acoustics boundary conditions

Solid wall (reflecting) boundary conditions:

$$u(a,t) = 0, \qquad u(b,t) = 0.$$

which can be written in terms of characteristic variables as:

$$w^2(a,t) = -w^1(a,t), \qquad w^1(b,t) = -w^2(a,t)$$

since $u = w^1 + w^2$.

$$q(a,t) = w^1(a,t)\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + w^2(a,t)\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p(a,t) \\ u(a,t) \end{bmatrix} = \begin{bmatrix} (-w^1(a,t) + w^2(a,t))Z_0 \\ w^1(a,t) + w^2(a,t) \end{bmatrix} = \begin{bmatrix} -2w^1(a,t)Z_0 \\ 0 \end{bmatrix}.$$

Figure 3.1

Figure 3.1 illustrates the acoustics solution with $u(x, 0) \equiv 0$.

An animation can be found in the **Clawpack Gallery**

Gallery of fvmbook applications $\longrightarrow$ Chapter 3

$\longrightarrow$ **animation of Pressure and Velocity**

Shows solution computed numerically on a fine grid, with:

- Solid wall boundary condition at the left,
- Outflow boundary condition at the right.