# Chapter 6 Exercises

From: *Finite Difference Methods for Ordinary and Partial Differential Equations*
by R. J. LeVeque, SIAM, 2007. `http://www.amath.washington.edu/~rjl/fdmbook`

**Exercise 6.1** *(Lipschitz constant for a one-step method)*

For the one-step method (6.17) show that the Lipschitz constant is $L' = L + \frac{1}{2}kL^2$.

**Exercise 6.2** *(Improved convergence proof for one-step methods)*

The proof of convergence of 1-step methods in Section 6.3 shows that the global error goes to zero as $k \to 0$. However, this bound may be totally useless in estimating the actual error for a practical calculation.

For example, suppose we solve $u' = -10u$ with $u(0) = 1$ up to time $T = 10$, so the true solution is $u(T) = e^{-100} \approx 3.7 \times 10^{-44}$. Using forward Euler with a time step $k = 0.01$, the computed solution is $U^N = (.9)^{100} \approx 2.65 \times 10^{-5}$, and so $E^N \approx U^N$. Since $L = 10$ for this problem, the error bound (6.16) gives

$$\|E^N\| \leq e^{100} \cdot 10 \cdot \|\tau\|_\infty \approx 2.7 \times 10^{44} \|\tau\|_\infty. \tag{E6.2a}$$

Here $\|\tau\|_\infty = |\tau^0| \approx 50k$, so this upper bound on the error does go to zero as $k \to 0$, but obviously it is not a realistic estimate of the error. It is too large by a factor of about $10^{50}$.

The problem is that the estimate (6.16) is based on the Lipschitz constant $L = |\lambda|$, which gives a bound that grows exponentially in time even when the true and computed solutions are decaying exponentially.

(a) Determine the computed solution and error bound (6.16) for the problem $u' = 10u$ with $u(0) = 1$ up to time $T = 10$. Note that the error bound is the same as in the case above, but now it is a reasonable estimate of the actual error.

(b) A more realistic error bound for the case where $\lambda < 0$ can be obtained by rewriting (6.17) as

$$U^{n+1} = \Phi(U^n)$$

and then determining the Lipschitz constant for the function $\Phi$. Call this constant $M$. Prove that if $M \leq 1$ and $E^0 = 0$ then

$$|E^n| \leq T\|\tau\|_\infty$$

for $nk \leq T$, a bound that is similar to (6.16) but without the exponential term.

(c) Show that for forward Euler applied to $u' = \lambda u$ we can take $M = |1 + k\lambda|$. Determine $M$ for the case $\lambda = -10$ and $k = 0.01$ and use this in the bound from part (b). Note that this is much better than the bound (E6.2a).

**Exercise 6.3** *(consistency and zero-stability of LMMs)*

Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

(a) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$

(b) $U^{n+1} = U^n$

(c) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$

(d) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

**Exercise 6.4** *(Solving a difference equation)*

Consider the difference equation $U^{n+2} = U^n$ with starting values $U^0$ and $U^1$. The solution is clearly

$$U^n = \begin{cases} U^0 & \text{if } n \text{ is even,} \\ U^1 & \text{if } n \text{ is odd.} \end{cases}$$

Using the roots of the characteristic polynomial and the approach of Section 6.4.1, another representation of this solution can be found:

$$U^n = (U^0 + U^1) + (U^0 - U^1)(-1)^n.$$

Now consider the difference equation $U^{n+4} = U^n$ with four starting values $U^0,\ U^1,\ U^2,\ U^3$. Use the roots of the characteristic polynomial to find an analogous represenation of the solution to this equation.

**Exercise 6.5** *(Solving a difference equation)*

(a) Determine the general solution to the linear difference equation $2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = 0$.

**Hint:** One root of the characteristic polynomial is at $\zeta = 1$.

(b) Determine the solution to this difference equation with the starting values $U^0 = 11$, $U^1 = 5$, and $U^2 = 1$. What is $U^{10}$?

(c) Consider the LMM

$$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = k(\beta_0 f(U^n) + \beta_1 f(U^{n+1})).$$

For what values of $\beta_0$ and $\beta_1$ is local truncation error $\mathcal{O}(k^2)$?

(d) Suppose you use the values of $\beta_0$ and $\beta_1$ just determined in this LMM. Is this a convergent method?

**Exercise 6.6** *(Solving a difference equation)*

(a) Find the general solution of the linear difference equation

$$U^{n+2} - U^{n+1} + 0.25U^n = 0.$$

(b) Determine the particular solution with initial data $U^0 = 2,\ U^1 = 3$. What is $U^{10}$?

(c) Consider the iteration

$$\begin{bmatrix} U^{n+1} \\ U^{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.25 & 1 \end{bmatrix} \begin{bmatrix} U^n \\ U^{n+1} \end{bmatrix}.$$

The matrix appearing here is the "companion matrix" (15.19) for the above difference equation. If this matrix is called $A$, then we can determine $U^n$ from the starting values using the $n$th power of this matrix. Compute $A^n$ as discussed in Section 15.2 and show that this gives the same solution found in part (b).

**Exercise 6.7** *(Convergence of backward Euler method)*

Suppose the function $f(u)$ is Lipschitz continuous over some domain $|u-\eta| \le a$ with Lipschitz constant $L$. Let $g(u) = u - kf(u)$ and let $\Phi(v) = g^{-1}(v)$, the inverse function.

Show that for $k < 1/L$, the function $\Phi(v)$ is Lipschitz continuous over some domain $|v - f(\eta)| \le b$ and determine a Lipschitz constant.

**Hint:** Suppose $v = u - kf(u)$ and $v^* = u^* - kf(u^*)$ and obtain an upper bound on $|u - u^*|$ in terms of $|v - v^*|$.

**Note:** The backward Euler method (5.21) takes the form

$$U^{n+1} = \Phi(U^n)$$

and so this shows that the implicit backward Euler method is convergent.

**Exercise 6.8** *(Fibonacci sequence)*

A Fibonacci sequence is generated by starting with $F_0 = 0$ and $F_1 = 1$ and summing the last two terms to get the next term in the sequence, so $F_{n+1} = F_n + F_{n-1}$.

(a) Show that for large $n$ the ratio $F_n/F_{n-1}$ approaches the "golden ratio" $\phi \approx 1.618034$.

(b) Show that the result of part (a) holds if any two integers are used as the starting values $F_0$ and $F_1$, assuming they are not both zero.

(c) Is this true for all real starting values $F_0$ and $F_1$ (not both zero)?