Chapter 4 Exercises

From: *Finite Difference Methods for Ordinary and Partial Differential Equations*
by R. J. LeVeque, SIAM, 2007. `http://www.amath.washington.edu/~rjl/fdmbook`

Exercise 4.1 *(Convergence of SOR)*

The m-file `iter_bvp_Asplit.m` implements the Jacobi, Gauss-Seidel, and SOR matrix splitting methods on the linear system arising from the boundary value problem $u''(x) = f(x)$ in one space dimension.

(a) Run this program for each method and produce a plot similar to Figure 4.2.

(b) The convergence behavior of SOR is very sensitive to the choice of ω (`omega` in the code). Try changing from the optimal ω to $\omega = 1.8$ or 1.95.

(c) Let $g(\omega) = \rho(G(\omega))$ be the spectral radius of the iteration matrix G for a given value of ω. Write a program to produce a plot of $g(\omega)$ for $0 \le \omega \le 2$.

(d) From equations (4.22) one might be tempted to try to implement SOR as

```
for iter=1:maxiter
   uGS = (DA - LA) \ (UA*u + rhs);
   u = u + omega * (uGS - u);
   end
```

where the matrices have been defined as in `iter_bvp_Asplit.m`. Try this computationally and observe that it does not work well. Explain what is wrong with this and derive the correct expression (4.24).

Exercise 4.2 *(Forward vs. backward Gauss-Seidel)*

(a) The Gauss-Seidel method for the discretization of $u''(x) = f(x)$ takes the form (4.5) if we assume we are marching forwards across the grid, for $i = 1,\ 2,\ \ldots,\ m$. We can also define a *backwards Gauss-Seidel method* by setting

$$u_i^{[k+1]} = \frac{1}{2}(u_{i-1}^{[k]} + u_{i+1}^{[k+1]} - h^2 f_i), \qquad \text{for } i = m,\ m-1,\ m-2,\ \ldots,\ 1. \tag{E4.2a}$$

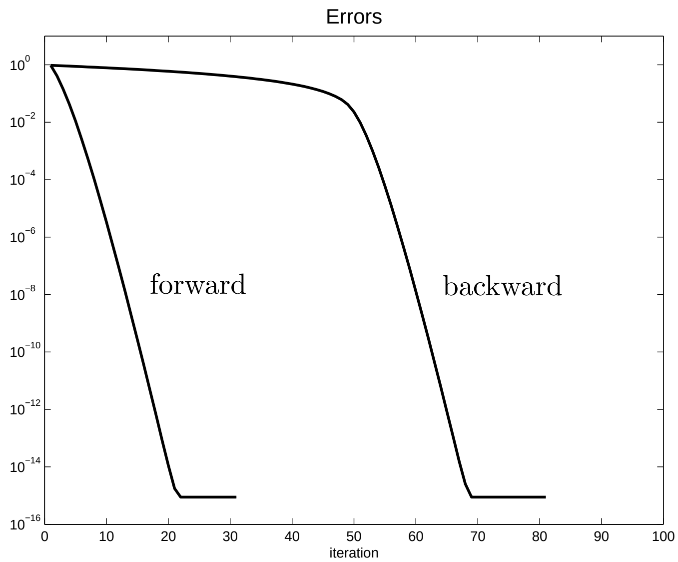
Show that this is a matrix splitting method of the type described in Section 4.2 with $M = D - U$ and $N = L$.

(b) Implement this method in `iter_bvp_Asplit.m` and observe that it converges at the same rate as forward Gauss-Siedel for this problem.

(c) Modify the code so that it solves the boundary value problem

$$\epsilon u''(x) = au'(x) + f(x), \qquad 0 \le x \le 1, \tag{E4.2b}$$

with $u(0) = 0$ and $u(1) = 0$, where $a \geq 0$ and the $u'(x_i)$ term is discretized by the one-sided approximation $(U_i - U_{i-1})/h$. Test both forward and backward Gauss-Seidel for the resulting linear system. With $a = 1$ and $\epsilon = 0.0005$. You should find that they behave very differently:



Explain intuitively why sweeping in one direction works so much better than in the other.

Hint: Note that this equation is the steady equation for an advection-diffusion PDE $u_t(x,t) + au_x(x,t) = \epsilon u_{xx}(x,t) - f(x)$. You might consider how the methods behave in the case $\epsilon = 0$.