# Chapter 2 Exercises

From: *Finite Difference Methods for Ordinary and Partial Differential Equations*
by R. J. LeVeque, SIAM, 2007. `http://www.amath.washington.edu/~rjl/fdmbook`

### Exercise 2.1 *(inverse matrix and Green's functions)*

(a) Write out the $5 \times 5$ matrix $A$ from (2.43) for the boundary value problem $u''(x) = f(x)$ with $u(0) = u(1) = 0$ for $h = 0.25$.

(b) Write out the $5 \times 5$ inverse matrix $A^{-1}$ explicitly for this problem.

(c) If $f(x) = x$, determine the discrete approximation to the solution of the boundary value problem on this grid and sketch this solution and the five Green's functions whose sum gives this solution.

### Exercise 2.2 *(Green's function with Neumann boundary conditions)*

(a) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e, find the function $G(x, \bar{x})$ solving

$$u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0$$

and the functions $G_0(x)$ solving

$$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$

and $G_1(x)$ solving

$$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$

(b) Using this as guidance, find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the $5 \times 5$ matrices $A$ and $A^{-1}$ for the case $h = 0.25$.

### Exercise 2.3 *(solvability condition for Neumann problem)*

Determine the null space of the matrix $A^T$, where $A$ is given in equation (2.58), and verify that the condition (2.62) must hold for the linear system to have solutions.

### Exercise 2.4 *(boundary conditions in `bvp` codes)*

(a) Modify the m-file `bvp2.m` so that it implements a Dirichlet boundary condition at $x = a$ and a Neumann condition at $x = b$ and test the modified program.

(b) Make the same modification to the m-file `bvp4.m`, which implements a fourth order accurate method. Again test the modified program.

**Exercise 2.5** *(accuracy on nonuniform grids)*

In Example 1.4 a 3-point approximation to $u''(x_i)$ is determined based on $u(x_{i-1})$, $u(x_i)$, and $u(x_{i+1})$ (by translating from $x_1$, $x_2$, $x_3$ to general $x_{i-1}$, $x_i$, and $x_{i+1}$). It is also determined that the truncation error of this approximation is $\frac{1}{3}(h_{i-1}-h_i)u'''(x_i)+O(h^2)$, where $h_{i-1} = x_i - x_{i-1}$ and $h_i = x_{i+1} - x_i$, so the approximation is only first order accurate in $h$ if $h_{i-1}$ and $h_i$ are $O(h)$ but $h_{i-1} \neq h_i$.

The program `bvp2.m` is based on using this approximation at each grid point, as described in Example 2.3. Hence on a nonuniform grid the local truncation error is $O(h)$ at each point, where $h$ is some measure of the grid spacing (e.g., the average spacing on the grid). If we assume the method is stable, then we expect the global error to be $O(h)$ as well as we refine the grid.

(a) However, if you run `bvp2.m` you should observe second-order accuracy, at least provided you take a smoothly varying grid (e.g., set `gridchoice = 'rtlayer'` in `bvp2.m`). Verify this.

(b) Suppose that the grid is defined by $x_i = X(z_i)$ where $z_i = ih$ for $i = 0,\ 1,\ \ldots,\ m+1$ with $h = 1/(m+1)$ is a uniform grid and $X(z)$ is some smooth mapping of the interval $[0,1]$ to the interval $[a,b]$. Show that if $X(z)$ is smooth enough, then the local truncation error is in fact $O(h^2)$. Hint: $x_i - x_{i-1} \approx hX'(x_i)$.

(c) What average order of accuracy is observed on a random grid? To test this, set `gridchoice = 'random'` in `bvp2.m` and increase the number of tests done, e.g., by setting `mvals = round(logspace(1,3,50));` to do 50 tests for values of $m$ between 10 and 1000.

**Exercise 2.6** *(ill-posed boundary value problem)*

Consider the following linear boundary value problem with Dirichlet boundary conditions:

$$
\begin{aligned}
&u''(x) + u(x) = 0 \quad \text{for } a < x < b \\
&u(a) = \alpha, \quad u(b) = \beta.
\end{aligned}
$$

Note that this equation arises from a linearized pendulum, for example.

(a) Modify the m-file `bvp2.m` to solve this problem. Test your modified routine on the problem with

$$a = 0, \quad b = 1, \quad \alpha = 2, \quad \beta = 3.$$

Determine the exact solution for comparison.

(b) Let $a = 0$ and $b = \pi$. For what values of $\alpha$ and $\beta$ does this boundary value problem have solutions? Sketch a family of solutions in a case where there are infinitely many solutions.

(c) Solve the problem with

$$a = 0, \quad b = \pi, \quad \alpha = 1, \quad \beta = -1.$$

using your modified `bvp2.m`. Which solution to the boundary value problem does this appear to converge to as $h \to 0$? Change the boundary value at $b = \pi$ to $\beta = 1$. Now how does the numerical solution behave as $h \to 0$?

(d) You might expect the linear system in part (c) to be singular since the boundary value problem is not well posed. It is not, because of discretization error. Compute the eigenvalues of the matrix $A$ for this problem and show that an eigenvalue approaches 0 as $h \to 0$. Also show that $\|A^{-1}\|_2$ blows up as $h \to 0$ so that the discretization is unstable.

**Exercise 2.7** *(nonlinear pendulum)*

(a) Write a program to solve the boundary value problem for the nonlinear pendulum as discussed in the text. See if you can find yet another solution for the boundary conditions illustrated in Figures 2.4 and 2.5.

(b) Find a numerical solution to this BVP with the same general behavior as seen in Figure 2.5 for the case of a longer time interval, say $T = 20$, again with $\alpha = \beta = 0.7$. Try larger values of $T$. What does $\max_i \theta_i$ approach as $T$ is increased? Note that for large $T$ this solution exhibits "boundary layers".