AMath 586 / ATM 581
Homework #1
Due Friday, April 6, 2007

Exercises from: *Finite Difference Methods for Ordinary and Partial Differential Equations* by R. J. LeVeque, SIAM, 2007. `http://www.amath.washington.edu/~rjl/fdmbook`

Exercise 5.1 *(Uniqueness for an ODE)*

Prove that the ODE

$$u'(t) = \frac{1}{t^2 + u(t)^2}, \quad \text{for } t \geq 1$$

has a unique solution for all time from any initial value $u(1) = \eta$.

Exercise 5.3 *(Lipschitz constant for a system of ODEs)*

Consider the system of ODEs

$$\begin{aligned} u_1' &= 3u_1 + 4u_2, \\ u_2' &= 5u_1 - 6u_2. \end{aligned}$$

Determine the Lipschitz constant for this system in the max-norm $\|\cdot\|_\infty$ and the 1-norm $\|\cdot\|_1$. (See Section 12.3.)

Exercise 5.4 *(Duhamel's principle)*

Check that the solution $u(t)$ given by (5.8) satisfies the ODE (5.6) and initial condition. Hint: To differentiate the matrix exponential you can differentiate the Taylor series (15.31) term by term.

Exercise 5.5 *(matrix exponential form of solution)*

The initial value problem

$$v''(t) = -4v(t), \qquad v(0) = v_0, \quad v'(0) = v_0'$$

has the solution $v(t) = v_0 \cos(2t) + \frac{1}{2} v_0' \sin(2t)$. Determine this solution by rewriting the ODE as a first order system $u' = Au$ so that $u(t) = e^{At}u(0)$ and then computing the matrix exponential using (15.30).

Exercise 5.8 *(Use of `ode113` and `ode45`)*

This problem can be solved by a modifying the m-files odesample.m and odesampletest.m available from the webpage.

Consider the third order initial value problem

$$\begin{aligned} &v'''(t) + v''(t) + 4v'(t) + 4v(t) = 4t^2 + 8t - 10, \\ &v(0) = -3, \quad v'(0) = -2, \quad v''(0) = 2. \end{aligned}$$

(a) Verify that the function

$$v(t) = -\sin(2t) + t^2 - 3$$

is a solution to this problem. How do you know it is the unique solution?

(b) Rewrite this problem as a first order system of the form $u'(t) = f(u(t), t)$ where $u(t) \in \mathbb{R}^3$. Make sure you also specify the initial condition $u(0) = \eta$ as a 3-vector.

(c) Use the MATLAB function `ode113` to solve this problem over the time interval $0 \le t \le 2$. Plot the true and computed solutions to make sure you've done this correctly.

(d) Test the MATLAB solver by specifying different tolerances spanning several orders of magnitude. Create a table showing the maximum error in the computed solution for each tolerance and the number of function evaluations required to achieve this accuracy.

(e) Repeat part (d) using the MATLAB function `ode45`, which uses an embedded pair of Runge-Kutta methods instead of Adams-Bashforth-Moulton methods.

Exercise 5.9 *(truncation errors)*

Compute the leading term in the local truncation error of the following methods:

(a) the trapezoidal method (5.22),

(b) the 2-step BDF method (5.25),

(c) the Runge-Kutta method (5.30).

Exercise 5.10 *(Derivation of Adams-Moulton)*

Determine the coefficients β_0, β_1, β_2 for the third order, 2-step Adams-Moulton method. Do this in two different ways:

(a) Using the expression for the local truncation error in Section 5.9.1,

(b) Using the relation

$$u(t_{n+2}) = u(t_{n+1}) + \int_{t_{n+1}}^{t_{n+2}} f(u(s))\, ds.$$

Interpolate a quadratic polynomial $p(t)$ through the three values $f(U^n)$, $f(U^{n+1})$ and $f(U^{n+2})$ and then integrate this polynomial exactly to obtain the formula. The coefficients of the polynomial will depend on the three values $f(U^{n+j})$. It's easiest to use the "Newton form" of the interpolating polynomial and consider the three times $t_n = -k$, $t_{n+1} = 0$, and $t_{n+2} = k$ so that $p(t)$ has the form

$$p(t) = A + B(t + k) + C(t + k)t$$

where A, B, and C are the appropriate divided differences based on the data. Then integrate from 0 to k. (The method has the same coefficients at any time, so this is valid.)

Exercise 5.17 *(R(z) for the trapezoidal method)*

(a) Apply the trapezoidal method to the equation $u' = \lambda u$ and show that

$$U^{n+1} = \left(\frac{1 + z/2}{1 - z/2}\right) U^n,$$

where $z = \lambda k$.

(b) Let

$$R(z) = \frac{1 + z/2}{1 - z/2}.$$

Show that $R(z) = e^z + \mathcal{O}(z^3)$ and conclude that the one-step error of the trapezoidal method on this problem is $\mathcal{O}(k^3)$ (as expected since the method is second order accurate).

Hint: One way to do this is to use the "Neumann series" expansion

$$\frac{1}{1 - z/2} = 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots$$

and then multiply this series by $(1 + z/2)$. A more general approach to checking the accuracy of rational approximations to e^z is explored in the next exercises.

Exercise 5.18 *(R(z) for Runge-Kutta methods)*

Any r-stage Runge-Kutta method applied to $u' = \lambda u$ will give an expression of the form

$$U^{n+1} = R(z)U^n$$

where $z = \lambda k$ and $R(z)$ is a rational function, a ratio of polynomials in z each having degree at most r. For an explicit method $R(z)$ will simply be a polynomial of degree r and for an implicit method it will be a more general rational function.

Since $u(t_{n+1}) = e^z u(t_n)$ for this problem, we expect that a pth order accurate method will give a function $R(z)$ satisfying

$$R(z) = e^z + \mathcal{O}(z^{p+1}) \quad \text{as } z \to 0, \tag{E5.18a}$$

as discussed in the Remark on page 129. The rational function $R(z)$ also plays a role in stability analysis as discussed in Section 7.6.2.

One can determine the value of p in (E5.18a). by expanding e^z in a Taylor series about $z = 0$, writing the $\mathcal{O}(z^{p+1})$ term as

$$Cz^{p+1} + \mathcal{O}(z^{p+2}),$$

multiplying through by the denominator of $R(z)$, and then collecting terms. For example, for the trapezoidal method of Exercise 5.17,

$$\frac{1 + z/2}{1 - z/2} = \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right) + Cz^{p+1} + \mathcal{O}(z^{p+2})$$

gives

$$1+\frac{1}{2}z = \left(1-\frac{1}{2}z\right)\left(1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots\right)+Cz^{p+1}+\mathcal{O}(z^{p+2})$$
$$= 1+\frac{1}{2}z-\frac{1}{12}z^3+\cdots+Cz^{p+1}+\mathcal{O}(z^{p+2})$$

and so

$$Cz^{p+1} = \frac{1}{12}z^3+\cdots\,,$$

from which we conclude that $p = 2$.

(a) Let

$$R(z) = \frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}.$$

Determine p for this rational function as an approximation to e^z.

(b) Determine $R(z)$ and p for the backward Euler method.

(c) Determine $R(z)$ and p for the TR-BDF2 method (5.36).