

Name: Your name here

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see <https://canvas.uw.edu/courses/1271892/assignments/4825694>

Problem 1

Let $U = [U_0, U_1, \dots, U_m]^T$ be a vector of function values at equally spaced points on the interval $0 \leq x \leq 1$, and suppose the underlying function is periodic and smooth. Then we can approximate the first derivative u_x at all of these points by DU , where D is circulant matrix such as

$$D_- = \frac{1}{h} \begin{bmatrix} 1 & & & & -1 \\ -1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & -1 & 1 \end{bmatrix}, \quad D_+ = \frac{1}{h} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{bmatrix}$$

for first-order accurate one-sided approximations, or

$$D_0 = \frac{1}{2h} \begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

for a second-order accurate centered approximation. (These are illustrated for a grid with $m + 1 = 5$ unknowns and $h = 1/5$.)

The advection equation $u_t + au_x = 0$ on the interval $0 \leq x \leq 1$ with periodic boundary conditions gives rise to the MOL discretization $U'(t) = -aDU(t)$ where D is one of the matrices above.

- (a) Discretizing $U' = -aD_-U$ by forward Euler gives the first order upwind method

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n),$$

where the index i runs from 0 to m with addition of indices performed mod $m + 1$ to incorporate the periodic boundary conditions.

Suppose instead we discretize the MOL equation by the second-order Taylor series method,

$$U^{n+1} = U^n - akD_-U^n + \frac{1}{2}(ak)^2D_-^2U^n.$$

Compute D_-^2 and also write out the formula for U_j^n that results from this method.

- (b) How accurate is the method derived in part (a) compared to the Beam-Warming method, which is also a 3-point one-sided method?
- (c) Suppose we make the method more symmetric:

$$U^{n+1} = U^n - \frac{ak}{2}(D_+ + D_-)U^n + \frac{1}{2}(ak)^2D_+D_-U^n.$$

Write out the formula for U_j^n that results from this method. What standard method is this?

Problem 2

- (a) Produce a plot similar to those shown in Figure 10.1 for the upwind method (10.21) with the same values of $a = 1$, $h = 1/50$ and $k = 0.8h$ used in that figure.
- (b) Produce the corresponding plot if the one-sided method (10.22) is instead used with the same values of a , h , and k as in part (a). Comment on how this relates to what stability theory tells you about this method with these parameters.

Problem 3

The notebook `notebooks/Advection.ipynb` illustrates several numerical methods for the advection equation, with periodic boundary conditions.

Familiarize yourself with how this works by experimenting with different parameter choices. In particular understand how the periodic boundary conditions are implemented.

- (a) Modify the notebook to also implement the Beam-Warming method from Section 10.4.2. Check that you observe second order accuracy. Make sure you implement the correct version based on the sign of a . (Assume $a > 0$.)
- (b) Experiment with different stepsizes to test the expected stability limit of this method. What happens if it the time step is chosen to be right at the stability limit? Why?
- (c) Do the oscillations that develop trail behind the Gaussian peak or lead ahead of it? You should be able to observe both cases depending on how you choose the time step. Confirm that what you see agrees with what is expected from Example 10.11, and produce sample plots for each case.