AMath 586 / ATM 581
Homework #5
Due Thursday, May 26, 2016

Name: Your name here

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see https://canvas.uw.edu/courses/1038268/assignments/3279572

Problem 1

Let $U = [U_0,\ U_1,\ \dots,\ U_m]^T$ be a vector of function values at equally spaced points on the interval $0 \le x \le 1$, and suppose the underlying function is periodic and smooth. Then we can approximate the first derivative u_x at all of these points by DU, where D is circulant matrix such as

$$D_- = \frac{1}{h}\begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}, \qquad D_+ = \frac{1}{h}\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{bmatrix}$$

for first-order accurate one-sided approximations or

$$D_0 = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

for a second-order accurate centered approximation. (These are illustrated for a grid with $m + 1 = 5$ unknowns and $h = 1/5$.)

The advection equation $u_t + au_x = 0$ on the interval $0 \le x \le 1$ with periodic boundary conditions gives rise to the MOL discretization $U'(t) = -aDU(t)$ where D is one of the matrices above.

(a) Discretizing $U' = -aD_-U$ by forward Euler gives the first order upwind method

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n),$$

where the index i runs from 0 to m with addition of indices performed mod $m+1$ to incorporate the periodic boundary conditions.

Suppose instead we discretize the MOL equation by the second-order Taylor series method,

$$U^{n+1} = U^n - akD_-U^n + \frac{1}{2}(ak)^2 D_-^2 U^n.$$

Compute D_-^2 and also write out the formula for U_j^n that results from this method.

(b) How accurate is the method derived in part (a) compared to the Beam-Warming method, which is also a 3-point one-sided method?

(c) Suppose we make the method more symmetric:

$$U^{n+1} = U^n - \frac{ak}{2}(D_+ + D_-)U^n + \frac{1}{2}(ak)^2 D_+D_-U^n.$$

Write out the formula for U_j^n that results from this method. What standard method is this?

Problem 2

(a) Produce a plot similar to those shown in Figure 10.1 for the upwind method (10.21) with the same values of $a = 1$, $h = 1/50$ and $k = 0.8h$ used in that figure.

(b) Produce the corresponding plot if the one-sided method (10.22) is instead used with the same values of a, h, and k as in part (a). Comment on how this relates to what stability theory tells you about this method with these parameters.

Problem 3

The notebook `notebooks/JSAnimation_demo.ipynb` explains how the functions in the file `notebooks/JSAnimation_frametools.py` can be used to create animations of a time-dependent solution. Using these requires the package `JSAnimations`, which is already available if you are using SageMathCloud or the Anaconda installation on another computer for example. You could also install it if necessary from this repository, `https://github.com/jakevdp/JSAnimation`, which contains more details and other examples.

If you want to use the utility functions from a different directory, e.g. `hw5`, you should copy the file `notebooks/JSAnimation_frametools.py` to that directory so that you can import it properly.

If you cannot get JSAnimation working, you can do this homework without making animations, but it's a cool tool you might want to use for other purposes.

The notebook `notebooks/Lax-Wendroff_periodic.ipynb` implements the Lax-Wendroff method for advection with periodic boundary conditions. Familiarize yourself with how this works by experimenting with different parameter choices. In particular understand how the periodic boundary conditions are implemented.

(a) Create a modified notebook that implements the Beam-Warming method from Section 10.4.2. Check that you observe second order accuracy. Make sure you implement the correct version based on the sign of a. (A more general code might take a as an input parameter and implement both versions with some logic to decide which to use, but you don't need to do this.)

(b) Experiment with different stepsizes to test the expected stability limit of this method. (And modify the warning message that is printed to warn of instability, since the limit is different than for Lax-Wendroff!) Produce plots for cases just below and just above the stability limit. What happens if it the time step is chosen to be right at the stability limit? Why?

(c) Do the oscillations that develop trail behind the Gaussian peak or lead ahead of it? You should be able to observe both cases depending on how you choose the time step. Confirm that what you see agrees with what is expected from Example 10.11, and produce sample plots for each case.