AMath 586 / ATM 581
Homework #3
Due Thursday, April 28, 2016

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/1038268/assignments/3262739`

Problem 1

Consider the implicit Runge-Kutta method

$$\begin{aligned} U^* &= U^n + \frac{k}{2} f(U^*, t_n + k/2), \\ U^{n+1} &= U^n + k f(U^*, t_n + k/2). \end{aligned} \tag{1}$$

The first step is Backward Euler to determine an approximation to the value at the midpoint in time and the second step is the midpoint method using this value.

(a) Determine the order of accuracy of this method.

(b) Plot the region of absolute stability.

(c) Is this method A-stable? Is it L-stable?

Problem 2

Plot the stability region for the TR-BDF2 method (8.6). You can start with the code in the notebook `Stability_Regions_onestep.ipynb`.

Observe that the method is A-stable and show by analyzing $R(z)$ that it is also L-stable.

Problem 3

Let $g(x) = 0$ represent a system of s nonlinear equations in s unknowns, so $x \in \mathbb{R}^s$ and $g : \mathbb{R}^s \to \mathbb{R}^s$. A vector $\bar{x} \in \mathbb{R}^s$ is a *fixed point* of $g(x)$ if

$$\bar{x} = g(\bar{x}). \tag{2}$$

One way to attempt to compute $\bar{x}$ is with *fixed point iteration*: from some starting guess x^0, compute

$$x^{j+1} = g(x^j) \tag{3}$$

for $j = 0,\ 1,\ \ldots$.

(a) Show that if there exists a norm $\|\cdot\|$ such that $g(x)$ is Lipschitz continuous with constant $L < 1$ in a neighborhood of $\bar{x}$, then fixed point iteration converges from any starting value in this neighborhood. **Hint:** Subtract equation (2) from (3).

(b) Suppose $g(x)$ is differentiable and let $g'(x)$ be the $s \times s$ Jacobian matrix. Show that if the condition of part (a) holds then $\rho(g'(\bar{x})) < 1$, where $\rho(A)$ denotes the spectral radius of a matrix.

(c) Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make N correction iterations, i.e., we set

$$\begin{aligned}
&\hat{U}^0 = U^n + kf(U^n) \\
&\text{for } j = 0,\ 1,\ \ldots,\ N-1 \\
&\qquad \hat{U}^{j+1} = U^n + kf(\hat{U}^j) \\
&\qquad \text{end} \\
&U^{n+1} = \hat{U}^N.
\end{aligned}$$

Note that this can be interpreted as a fixed point iteration for solving the nonlinear equation

$$U^{n+1} = U^n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

Plot the stability region S_N of this method for $N = 2,\ 5,\ 10,\ 20,\ 50$ and observe that in fact the stability region does not grow much in size.

(d) Using the result of part (b), show that the fixed point iteration being used in the predictor-corrector method of part (c) can only be expected to converge if $|k\lambda| < 1$ for all eigenvalues λ of the Jacobian matrix $f'(u)$.

(e) Based on the result of part (d) and the shape of the stability region of Backward Euler, what do you expect the stability region S_N of part (c) to converge to as $N \to \infty$?

Problem 4

(a) Write Python code to compute the exact solution the linear system (7.10) modeling simple reaction kinetics. Reproduce Figure 7.4 by using the same choice of parameters and initial conditions as specified in Example 7.9.

(b) Write code to solve the same problem with the Forward Euler method and observe what happens if the time step k is chosen to be the largest possible value allowed by absolute stability. Check with smaller k also to verify that the method converges.

(c) Change the reaction rates to $K_1 = 10,\ K_2 = 1$ and repeat the tests of the previous part.

(d) Implement the backward Euler method and show that it remains stable (although perhaps not accurate) for arbitrarily large time steps.

(e) Implement the TR-BDF2 method on this same problem and verify that it is second-order accurate at time $T = 2$ with $K_1 = 10,\ K_2 = 1$ and the same initial conditions as before.