AMath 586 / ATM 581
Homework #5
Due Tuesday, May 26, 2015

Name: Your name here

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see https://canvas.uw.edu/courses/962872/assignments/2869615

Problem 1

Let $U = [U_0,\ U_1,\ \ldots,\ U_m]^T$ be a vector of function values at equally spaced points on the interval $0 \le x \le 1$, and suppose the underlying function is periodic and smooth. Then we can approximate the first derivative u_x at all of these points by DU, where D is circulant matrix such as

$$D_- = \frac{1}{h}\begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}, \qquad D_+ = \frac{1}{h}\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{bmatrix}$$

for first-order accurate one-sided approximations or

$$D_0 = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

for a second-order accurate centered approximation. (These are illustrated for a grid with $m + 1 = 5$ unknowns and $h = 1/5$.)

The advection equation $u_t + au_x = 0$ on the interval $0 \le x \le 1$ with periodic boundary conditions gives rise to the MOL discretization $U'(t) = -aDU(t)$ where D is one of the matrices above.

(a) Discretizing $U' = -aD_-U$ by forward Euler gives the first order upwind method

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n),$$

where the index i runs from 0 to m with addition of indices performed mod $m + 1$ to incorporate the periodic boundary conditions.

Suppose instead we discretize the MOL equation by the second-order Taylor series method,

$$U^{n+1} = U^n - akD_-U^n + \frac{1}{2}(ak)^2 D_-^2 U^n.$$

Compute D_-^2 and also write out the formula for U_j^n that results from this method.

(b) How accurate is the method derived in part (a) compared to the Beam-Warming method, which is also a 3-point one-sided method?

(c) Suppose we make the method more symmetric:

$$U^{n+1} = U^n - \frac{ak}{2}(D_+ + D_-)U^n + \frac{1}{2}(ak)^2 D_+ D_- U^n.$$

Write out the formula for U_j^n that results from this method. What standard method is this?

Problem 2

(a) Produce a plot similar to those shown in Figure 10.1 for the upwind method (10.21) with the same values of $a = 1$, $h = 1/50$ and $k = 0.8h$ used in that figure.

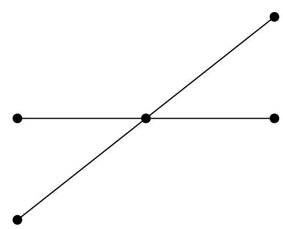
(b) Produce the corresponding plot if the one-sided method (10.22) is instead used with the same values of a, h, and k.

Problem 3

Suppose $a > 0$ and consider the following *skewed leapfrog* method for solving the advection equation $u_t + au_x = 0$:

$$U_j^{n+1} = U_{j-2}^{n-1} - \left(\frac{ak}{h} - 1\right)(U_j^n - U_{j-2}^n).$$

The stencil of this method is



Note that if $ak/h \approx 1$ then this stencil roughly follows the characteristic of the advection equation and might be expected to be more accurate than standard leapfrog. (If $ak/h = 1$ the method is exact.)

(a) What is the order of accuracy of this method?

(b) For what range of Courant number ak/h does this method satisfy the CFL condition?

(c) Show that the method is in fact stable for this range of Courant numbers by doing von Neumann analysis. **Hint:** Let $\gamma(\xi) = e^{i\xi h} g(\xi)$ and show that γ satisfies a quadratic equation closely related to the equation (10.34) that arises from a von Neumann analysis of the leapfrog method.

Problem 4

Derive the modified equation (10.45) for the Lax-Wendroff method.

Problem 5

The m-file `advection_LW_pbc.m` from the book repository implements the Lax-Wendroff method for the advection equation on $0 \le x \le 1$ with periodic boundary conditions.

The class repository contains a Python translation, `$AM586/codes/advection_LW_pbc.py`.

You might want to first experiment with these and see how Lax-Wendroff behaves for various grid resolutions. Note that the way this problem is set up, the solution advects twice around the domain over time 1 and should end up agreeing with the initial data.

(a) Modify one of these files to implement the leapfrog method and verify that this is second order accurate. Note that you will have to specify two levels of initial data. For the convergence test set $U_j^1 = u(x_j, k)$, the true solution at time k.

(b) Modify your code so that the initial data consists of a wave packet

$$\eta(x) = \exp(-\beta(x - 0.5)^2)\sin(\xi x)$$

Work out the true solution $u(x, t)$ for this data. Using $\beta = 100$, $\xi = 80$ and $U_j^1 = u(x_j, k)$, test that your code still exhibits second order accuracy for k and h sufficiently small.

(c) Using $\beta = 100$, $\xi = 150$ and $U_j^1 = u(x_j, k)$, estimate the group velocity of the wave packet computed with leapfrog using $m = 199$ and $k = 0.4h$. How well does this compare with the value (10.52) predicted by the modified equation?

Note: Early editions of the text book had a typo in (10.52). There should be a factor ν^2 in the denominator.