AMath 586 / ATM 581
Homework #1
Due Thursday, April 9, 2015

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/962872/assignments/2829773`

Problem 1

Prove that the ODE

$$u'(t) = \frac{1}{t^2 + u(t)^2}, \quad \text{for } t \geq 1$$

has a unique solution for all time from any initial value $u(1) = \eta$.

Problem 2

Consider the system of ODEs

$$u_1' = 3u_1 + 4u_2,$$
$$u_2' = 5u_1 - 6u_2.$$

Determine the best possible Lipschitz constant for this system in the max-norm $\|\cdot\|_\infty$ and the 1-norm $\|\cdot\|_1$. (See Appendix A.3.)

Problem 3

The initial value problem

$$v''(t) = -4v(t), \qquad v(0) = v_0, \quad v'(0) = v_0'$$

has the solution $v(t) = v_0 \cos(2t) + \frac{1}{2}v_0' \sin(2t)$. Determine this solution by rewriting the ODE as a first order system $u' = Au$ so that $u(t) = e^{At}u(0)$ and then computing the matrix exponential using (D.30) in Appendix D.

Problem 4

Compute the leading term in the local truncation error of the following methods:

(a) the trapezoidal method (5.22),

(b) the 2-step Adams-Bashforth method,

(c) the Runge-Kutta method (5.32).

Problem 5

Determine the coefficients $\beta_0,\ \beta_1,\ \beta_2$ for the third order, 2-step Adams-Moulton method. Do this in two different ways:

(a) Using the expression for the local truncation error in Section 5.9.1,

(b) Using the relation

$$u(t_{n+2}) = u(t_{n+1}) + \int_{t_{n+1}}^{t_{n+2}} f(u(s))\, ds.$$

Interpolate a quadratic polynomial $p(t)$ through the three values $f(U^n)$, $f(U^{n+1})$ and $f(U^{n+2})$ and then integrate this polynomial exactly to obtain the formula. The coefficients of the polynomial will depend on the three values $f(U^{n+j})$. It's easiest to use the "Newton form" of the interpolating polynomial and consider the three times $t_n = -k$, $t_{n+1} = 0$, and $t_{n+2} = k$ so that $p(t)$ has the form

$$p(t) = A + B(t + k) + C(t + k)t$$

where A, B, and C are the appropriate divided differences based on the data. Then integrate from 0 to k. (The method has the same coefficients at any time, so this is valid.)

Problem 6

The initial value problem

$$\begin{aligned} u'(t) &= u(t)^2 - \sin(t) - \cos^2(t), \\ u(0) &= 1 \end{aligned} \tag{1}$$

has the solution $u(t) = \cos(t)$.

Write a computer code (preferably in Python or Matlab) to solve problem (1) up to time $T = 8$ with various different time steps $\Delta t = T/N$, with

$$N = 25,\ 50,\ 100,\ 200,\ 400,\ 800,\ 1600,\ 3200.$$

Do this using two different methods:

(a) Forward Euler

(b) The Runge-Kutta method (5.32). Note that this should be second order accurate for sufficiently small Δt. If not, then you might have a bug.

Produce a log-log plot of the errors versus Δt, with both plots in the same figure. Figure 1 below shows how this might look for Forward Euler.

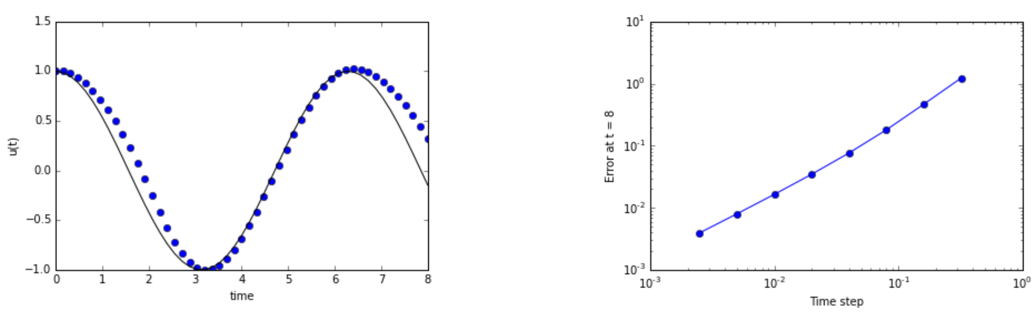

Figure 1: Left: the Euler solution with $N = 50$, Right: Log-log plot of the error in Forward Euler.