Name: Your name here

This exam is due to Canvas by 11:00pm PDT on the due date.

To submit, see https://canvas.uw.edu/courses/962872/assignments/2878942

Please work on this exam on your own! You may consult the literature and use the discussion board and office hours if questions arise.

Problem 1

Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}).$$ (1)

for the advection equation $u_t + au_x = 0$ on $0 \leq x \leq 1$ with periodic boundary conditions, on a grid with spacing $h = 1/(m+1)$ for some integer m.

(a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A? Don't forget the boundary conditions.

(b) Determine the eigenvalues of the matrix A. **Hint:** the eigenvectors are the same as for similar matrices you have seen.

(c) Suppose we want to fix the Courant number ak/h as $k, h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix A and the stability region of the trapezoidal method.

(d) For what range of ak/h will the CFL condition be satisfied for this method (with periodic boundary conditions)? Consider both $a > 0$ and $a < 0$. Is satisfying the CFL condition sufficient for stability in all cases?

Problem 2

Consider the dispersive PDE $u_t + \gamma u_{xxx} = 0$ with periodic boundary conditions on an interval $a \leq x \leq b$.

(a) The third-derivative term could be approximated by either

$$u_{xxx}(x_j, t_n) \approx \frac{1}{h^3}(-U_{j-2}^n + 3U_{j-1}^n - 3U_j^n + U_{j+1}^n)$$ (2)

or by

$$u_{xxx}(x_j, t_n) \approx \frac{1}{h^3}(-U_{j-1}^n + 3U_j^n - 3U_{j+1}^n + U_{j+2}^n)$$ (3)

Show that either one is first-order accurate.

(b) Suppose you plan to use a trapezoid discretization in time for a problem where $\gamma > 0$. Which form of spatial discretization (2) or (3) would be better? Justify your answer.

(c) Implement a Crank-Nicolson type method for this equation, which uses the trapezoid method in time together with the spatial discretization from above that you determined is best for the case

$\gamma > 0$. Your implementation should allow arbitrary values of the endpoints a and b of the spatial interval, since two different choices are required below.

Test your method by showing it is first order accurate for initial data $u(x, 0) = \sin(px)$ for integer p such as $p = 2$ on the interval $0 \le x \le 2\pi$. Note that you can compute the exact solution for such data.

Problem 3

Now consider the KdV equation $u_t + 6uu_x + u_{xxx} = 0$. This Korteweg - de Vries equation is famous — read about it if you are not familiar with it. Even though this equation is nonlinear, there are exact "solitary wave" solutions that consist of a translating wave of the form $u(x, t) = \eta(x - \alpha^2 t)$ for special initial data of the form

$$\eta(x) = \frac{\alpha^2}{2} \operatorname{sech}^2 (\alpha x/2) \tag{4}$$

for any choice of parameter $\alpha > 0$. Note that taller pulses propagate more rapidly. An amazing feature of such problems is the way two such pulses interact as "solitons".

Construct a first-order accurate method by combining your solution to Problem 2 with a first-order upwind method for the hyperbolic equation

$$u_t + 3(u^2)_x = 0. \tag{5}$$

Use the first-order fractional step approach to combine the methods — i.e. in each time step, first advance the solution by taking a step with the $u_t + u_{xxx} = 0$ to get an intermediate solution U^*, and then use this as initial data for a step in which you solve (5) by the upwind method for this nonlinear conservation law.

Again use periodic boundary conditions. The true solution is not periodic, but decays exponentially to zero so as long as the interval is long enough relative to the width of the solitary wave, so this should not affect errors computed using a periodic extension of (4).

Test your method on the interval $-1 \le x \le 3$ with $\alpha = 10$ out to time $t = 0.02$. Demonstrate that your method is stable and first-order accurate by refining the grid with $r = k/h$ fixed.

Estimate the maximum stable value of r based on stability theory, justifying your estimate, and confirm that this is consistent with what you observe from your numerical method.

Note that this is not a very accurate method! If you want to play around with soliton interaction (optional) you might try the following data:

$$\eta(x) = \frac{\alpha_1^2}{2} \operatorname{sech}^2 (\alpha_1(x + 2)/2) + \frac{\alpha_2^2}{2} \operatorname{sech}^2 (\alpha_2(x + 1)/2) \tag{6}$$

with $\alpha_1 = 25$, $\alpha_2 = 16$ on the interval $-3 \le x \le 2$.

If you want to read about (and try out) a much more accurate method for this problem, see for example Program 27 in Chapter 10 of "Spectral Methods in Matlab" by L. N. Trefethen (SIAM 2000), available as an ebook at http://epubs.siam.org/doi/book/10.1137/1.9780898719598. The programs are available at http://people.maths.ox.ac.uk/trefethen/spectral.html.