AMath 585
Homework #3
Due Thursday, February 6, 2020

Homework is due to Canvas by 11:00pm PDT on the due date.

To submit, see `https://canvas.uw.edu/courses/1352870/assignments/5237966`

Problem 1. Consider the nonlinear boundary value problem

$$\sin(u''(x)) = u(x)\exp(u'(x)) + f(x)$$

for $0 \le x \le 1$, with Dirichlet boundary conditions $u(0) = \alpha,\ u(1) = \beta$.

(a) Discretize using the standard second-order centered approximations for $u'(x_i)$ and $u''(x_i)$, giving a nonlinear system of equation $G(U) = 0$ where $G : \mathbb{R}^m \to \mathbb{R}^m$ and U is the vector of interior unknowns. What is the i'th component $G_i(U)$?

(b) What is the (i, j) element of the Jacobian matix $G'(U)$ needed to implement Newton's method for this system?

(c) Is the Jacobian matrix symmetric in general?

Problem 2.

Consider the linearized pendulum problem of Section 2.16, which we will rewrite with x in place of t and now calling the angle $u(x)$ for consistency with our other BVPs:

$$u''(x) = -\gamma^2 u(x)$$

on the interval $0 \le x \le 1$, with $\gamma^2 = g/L$.

(a) Confirm that the general solution to this ODE is $u(x) = c_1 \sin(\gamma x) + c_2 \cos(\gamma x)$.

(b) If we impose Dirichlet boundary conditions $u(0) = \alpha$ and $u(1) = \beta$, show that this leads to a 2×2 linear system of equations to determine the coefficients c_1 and c_2. Solve this system for the case $\gamma = \pi/2$ to find c_1 and c_2 in terms of α and β.

(c) Show that if $\gamma = \pi$ then the system for c_1, c_2 is singular. For what choices of α, β does the system have a solution? In such cases it has infinitely many solutions, what are they?

(d) Recall that $-\pi^2$ is an eigenvalue of the operator ∂_x^2 on the interval $[0, 1]$ with homogeneous boundary conditions $u(0) = u(1) = 0$. The linear pendulum equation can be written as $\mathcal{L}u = 0$, where $L = \partial_x^2 + \gamma^2$. What are the eigenvalues and eigenfunctions of this operator and how do these relate to your answer to part (c)?

(e) Suppose we discretize this BVP with the usual second-order centered approximation, giving a tridiagonal matrix A for the system for interior unknowns $[U_1,\ \ldots,\ U_m]$. What are the eigenvalues and eigenvectors of this matrix? Hint: if T is the tridiagonal matrix of (2.10) then $A = T + \gamma^2 I$.

(f) Note that if $\gamma = \pi$ then this matrix is not singular (all the eigenvalues are nonzero) for $h > 0$, so the discrete system has a unique solution for any $h > 0$. However, we expect trouble as $h \to 0$, since we are approximating a BVP that does not have a unique solution for this choice of γ. Show that in the 2-norm the method is *not stable* in the sense of Definition 2.1 in this case. How rapidly does $\|A^{-1}\|_2$ grows as $h \to 0$. (E.g. like $1/h$? or $1/h^2$?)

(g) On the other hand, for any γ that is not an integer multiple of π, show that this method is stable in the 2-norm.

Problem 3.

If $A \in \mathbb{R}^{m \times m}$ and $u \in \mathbb{R}^m$ is any nonzero vector, then the scalar value $Q(u) = u^T Au/u^T u$ is called the Rayleigh quotient.

(a) Show that if v is an eigenvector of A then $Q(v) = \lambda$, the corresponding eigenvalue.

(b) If $A = A^T$ is a symmetric matrix then the eigenvalues must be real. The matrix is called *symmetric positive definite* if the eigenvalues of A are all positive, or *symmetric negative definite* if they are all negative.

Show that if A is symmetric positive or negative definite then

$$\min_p \lambda_p \leq Q(u) \leq \max_p \lambda_p. \tag{1}$$

Hint: Recall that if A is symmetric then it is diagonalizable and the eigenvectors are mutually orthogonal, so we can write $A = V\Lambda V^T$ where V is the matrix of normalized eigenvectors (each column v_j has 2-norm equal to 1). So V is an "orthogonal matrix" with $V^{-1} = V^T$. Hence any vector $u \in \mathbb{R}^m$ can be written as $u = Vy$ for $y = V^T u$. (See also Appendix C.)

Conversely, show that if A is symmetric and if there are constants C_1, C_2 such that $C_1 \leq Q(u) \leq C_2$ for all nonzero vectors u, then the eigenvalues of A all satisfty $C_1 \leq \lambda \leq C_2$.

Investigating $Q(u)$ can sometimes help us to show that the eigenvalues of A are bounded away from 0, which can be useful in proving stability of a method.

The next problem illustrates one such case.

Problem 4.

Suppose we discretize the *nonlinear* pendulum problem $u''(x) = -\gamma^2 \sin(u(x))$ (again with Dirichlet boundary conditions) using the standard second order centered approximation to $u''(x_i)$, as discussed in Section 2.16.

The Jacobian matrix $G'(U)$ now has the form $G'(U) = T + D$, where T is the standard tridiagonal matrix of (2.10) and D is now a diagonal matrix that is no longer a scalar multiple of the identity matrix. It is similar to (2.82) but with an additional factor γ^2.

We can no longer compute the eigenvalues or eigenvectors of $A = T + D$ exactly.

Consider a problem for which $\gamma < \pi$ and suppose we also know there is a unique exact solution $u(x)$ to the boundary value problem being considered (there is in this case, but you don't need to prove it).

Then, using the results of Problem 3, show that as $h \to 0$ we can find a uniform bound on $\|(G'(U))^{-1}\|_2$ and hence this method for the nonlinear pendulum is stable in the sense of Definition 2.2.