

# Distributed Learning in Network Games: a Dual Averaging Approach

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**Abstract**—In this paper, we propose a distributed no-regret learning algorithm for network games using a primal-dual method, i.e., dual averaging. With only locally available observations, we consider the scenario where each player optimizes a global objective, formed by local objective functions on the nodes of a given communication graph. Our learning algorithm for each player involves taking steps along their individual pay-off gradients, dictated by local observations of the other player’s actions—where the local nature of information exchange is encoded by the network. The output is then projected back again locally to the set of admissible actions for each player. We show the convergence of this distributed learning algorithm for the case of a deterministic network that is subjected to two teams with distinct objectives. Our analysis indicates the key correlation between the rate of convergence and network structure/connectivity that also appeared in distributed optimization setups via dual averaging. Finally, illustrative examples showcase the performance of our algorithm in relation to the size and connectivity of the communication network.

**Keywords:** *Network games, distributed learning, dual averaging, Nash equilibrium*

## I. INTRODUCTION

Networked systems analysis has been on the cutting edge of multi-disciplinary research over the past few years with applications spanning from robotic swarms to biological networks. Common to all of these systems, a global objective is achieved based on local interactions which in turn require local decision-making that is inherently restricted by limited information exchange and prescribed set of admissible policies [1]. While provably effective, many established computationally efficient optimization algorithms [2]–[5] suffer from these limitations, as they become mostly intractable when the network grows large in scale. Not surprisingly, there has been an extensive literature on exploiting the structure of a communication graph to reduce the complexity of methods such as subgradient optimization [6], mirror decent and dual averaging [7], [8], and sequential decision-making [9] to name a few. However, while these works mainly focus on a cooperative information sharing, many of the real-world systems exhibit non-cooperative behaviors due to a plethora of reasons including intrusions/attacks, greedy agents, competition for constrained resources, misaligned

incentives, or the adversarial nature of environments that necessitate a game-theoretical model.

Game theory has been successfully employed in non-cooperative decision-making [10]–[12]. A variety of dynamic games including zero-sum, non-zero-sum [13] and Stackelberg [14] have been studied in the literature for decades that model various non-cooperative interactions among decision makers, e.g. network security [15], and wireless networks [16], [17], and multi-agent systems such as Pursuit Evasion games [18], [19],

Conventional centralized game-theoretic machinery are barely leveraged in real-world decision-making due to poor computational performance, lack of information, and policy limitations. In this direction, well-established algorithms such as primal-dual methods have been used for convergence analysis of learning and optimization on games. For instance, by introducing *variational stability* condition, [20] considers a centralized multiagent decision process where each agent employs an online mirror descent learning algorithm. Also, [21] studies the behaviour of learning with “bandit” feedback (a framework for extremely low-information environments) in non-cooperative concave games. However, in spite of the improved convergence rates, scalability and limited communications are inherent source of intractability in these learning setups.

As the key connection, Nash Equilibrium (NE) plays a central role in game theory roughly defined as a strategy from which no player has any incentive to deviate unilaterally. Upon the convexity and continuous differentiability, the NE can be characterized as the solution of a variational inequality (VI) problem. Distributed Nash seeking refers to the class of problems that aims for learning a global NE with only local information [22]–[25]. In this direction, each node in the network usually represents an individual player, e.g., [26] proposes an accelerated gradient decent for monotone games and [27] considers a zero-sum game between two networks characterizing conditions for convergence to NE. Additionally, the existence of a unique Nash equilibrium can be guaranteed under assumptions on the admissible action sets (compactness) and the game Jacobian (monotonicity and its variants) [28]–[30].

In this work, we introduce a non-cooperative game between two teams, each consisting of players that interact over a network. While the goal of each team is to learn the global NE, players have no information about the other side, neither do they enjoy a global decision-making capability. Thus the equilibrium learning is only based on local observations of the opponent actions and distributed decision-making of

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The research of the authors has been supported by NSF grant SES-1541025 and Air Force Office of Scientific Research under grant number FA9550-16-1-0022.

nodes within their team. This setup resembles a scenario where each node in the network is subjected to actions that contribute to distinct objectives—in this sense, there is duality in the interactions between nodes in the network. For example, each node can represent a socio-economic entity, with objectives that are not completely aligned with each other (altruistic vs. profit-seeking). Financial networks consisting of nodes that are subjected to political influence from two opposing parties yet provide another scenario of interest. The network in these scenarios provides the backbone for information exchange and coordination amongst the teams. As such, it becomes imperative to characterize the role of the network in the evolution of team’s strategies, potentially towards NE. Of prime interest in this work is how algebraic and combinatorial properties of the network, such as its connectivity, contribute to the convergence of distributed learning in the context of games. The choice of a primal/dual approach that is built around dual averaging [3], [4], in conjunction with a certain monotonicity assumption, is primarily motivated by this overarching objective; this choice is also consistent with how dual averaging has been used in the context of distributed optimization to underscore the “network-effect” on the convergence and optimality properties of distributed optimization, where there is no “duality” in the nodes’ operation. However, in order to extend the work of [8] to the game setting, one has to pay a particular attention to the information structure, as for example, it would be unreasonable to assume information sharing with the opponents in the game setting.

Accordingly, in our contribution: (1) We suppose players have neither *a priori* information about their adversary, nor global coordination over the network. Instead, they have a chance to choose a local strategy at each node and receive a local cost (reward), resulting in learning the global NE in a distributed fashion, (2) We propose intuitive assumptions which lead us to the convergence proof of objective values to those of NE. In addition, we will discuss the convergence of running average of actions to NE at each node under stronger regularity assumptions. It is shown that the algorithm enjoys a similar regret bound as dual averaging methods which is well-known to be tight in black-box setting [2].

The rest of the paper is organized as follows: In §II we provide a quick overview of mathematical tools that are used in the paper. In §III we introduce the problem setup and propose our method. In §IV we build up the required foundations to ultimately prove convergence of our algorithm to NE. We provide an illustrative example in §V and concluding remarks and future directions are addressed in §VI.

## II. MATHEMATICAL PRELIMINARIES

We denote by  $\mathbb{R}$  the set of real numbers. A column vector with  $n$  elements is referred to as  $v \in \mathbb{R}^n$ , where  $v_i$  represents the  $i$ th element in  $v$ . The matrix  $M \in \mathbb{R}^{p \times q}$  contains  $p$  rows and  $q$  columns with  $M_{ij}$  denoting the element in the  $i$ th row and  $j$ th column of  $M$ . The square matrix  $N \in \mathbb{R}^{n \times n}$  is *symmetric* if  $N^\top = N$ , where  $N^\top$

denotes the *transpose* of the matrix  $N$ . The  $n \times 1$  vector of all ones is denoted by  $\mathbb{1}$ . A doubly stochastic matrix  $P \in \mathbb{R}^{n \times n}$  is defined as a non-negative square matrix such that  $\sum_j P_{ij} = \sum_j P_{jk} = 1$  for all  $i$  and  $k$ . We define  $[n] = \{1, \dots, n\}$ . The *Euclidean norm* of a vector  $\mathbf{x} \in \mathbb{R}^n$  is defined as  $\|\mathbf{x}\|_2 = (\mathbf{x}^\top \mathbf{x})^{1/2} = (\sum_{i=1}^n x_i^2)^{1/2}$  and the *dual norm* to  $\|\cdot\|_2$  is denoted by  $\|x\|_* := \sup_{\|u\|_2=1} \langle v, u \rangle$ . Also, the *1-norm* is defined as  $\|x\|_1 = \sum_{i=1}^n |x_i|$ . A function  $f$  is convex if  $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$  for all  $\theta \in (0, 1)$  and for all  $x, y$  in its convex domain, and  $g$  is a sub-gradient of  $f$  at point  $z$  if  $f(y) \geq f(z) + g^\top(y - z)$  for all  $y$ . If  $f$  is convex and differentiable, the gradient of  $f$ ,  $\nabla f(x)$ , is also a sub-gradient of  $f$  at  $x$ . A graph is characterized by the 2-tuple  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  is the set of nodes and  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  denotes the set of edges. An edge exists from node  $i$  to  $j$  if  $(i, j) \in \mathcal{E}$  which can also be shown by  $j \in \mathcal{N}_i$  where  $\mathcal{N}_i$  is the set of neighbors of node  $i$ . Then we say a graph is *complete* if  $(i, j) \in \mathcal{E}$  for all nodes  $i, j$ , is *random  $k$ -regular* if  $|\mathcal{N}_i| = k$  for all  $i$  in some random order, and is called a *cycle* in case  $j \in \mathcal{N}_i$  if and only if  $j = i \pm 1$  with a cyclic order, i.e.,  $(1, n) \in \mathcal{E}$  in case there are  $n$  nodes in the graph.

## III. PROBLEM SETUP

In this section we introduce the main framework of our analysis. Herein, we briefly mention some background material on dual averaging which is the workhorse of our methodology. We then continue by proposing the distributed setup followed by a two-player game-theoretic framework, which can be generalized to multi-player setting with no extra effort.

### A. Standard Dual Averaging

The dual averaging algorithm proposed by Nesterov [3] is a subgradient scheme for non-smooth convex problems. The primal-dual nature of this method generates two sequence of iterates  $\{x(t), z(t)\}_{t=0}^\infty$  contained within  $\mathcal{X} \times \mathbb{R}^d$  such that the updates of  $z(t)$  is responsible for averaging the support functions in the dual space, while the updates of  $x(t)$  establishes a dynamically updated scale between the primal and dual spaces. More precisely, after receiving the subgradient  $g(t) \in \partial f(x(t))$  at iteration  $t$ , the algorithm is updated as follows,

$$\begin{aligned} z(t+1) &= z(t) + \gamma(t)g(t), \\ x(t+1) &= \Pi_{\mathcal{X}}^\psi(-z(t+1), \alpha(t)), \end{aligned} \quad (1)$$

where  $\gamma(t) > 0$ ,  $\{\alpha(t)\}_{t=0}^\infty$  is a positive non-increasing sequence, and

$$\Pi_{\mathcal{X}}^\psi := \operatorname{argmin}_{x \in \mathcal{X}} \left\{ \langle -z, x \rangle + \frac{1}{\alpha} \psi(x) \right\} \quad (2)$$

is a generalized projection according to a strongly convex *prox-function*  $\psi(\cdot)$ .

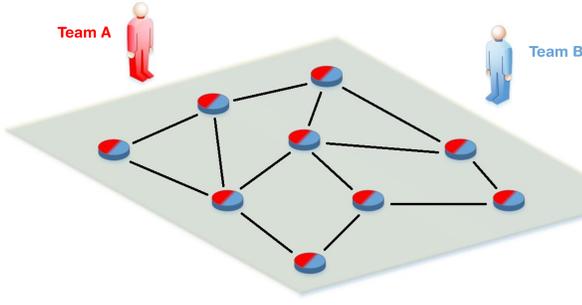


Fig. 1: Schematic of our game setup including two teams (red A and blue B) with each team having a representative at each node.

### B. Our Model

We consider the distributed learning problem for a game between two teams (players), both playing on a network consisting of  $n$  nodes connected via a communication graph  $\mathcal{G}$ . To this end, each team has a representative on each node, hence  $2n$  members in total (Figure 1). The teams are grouped within the sets  $\mathcal{I}_\ell = \{(\ell, 1), \dots, (\ell, n)\}$  for  $\ell \in \{A, B\}$  and each team has a choice of action  $x_\ell \in \mathcal{X}_\ell \subset \mathbb{R}^{d_\ell}$  that minimizes the following global cost:

$$f_\ell(x_A, x_B) = \frac{1}{n} \sum_{i=1}^n f_{\ell,i}(x_A, x_B), \quad \ell \in \{A, B\} \quad (3)$$

which is the average of its members' costs at each node  $i$ , denoted by  $f_{\ell,i}$ . The goal is to learn a global NE while players have no global decision-making capability. Instead,  $\mathcal{G}$  is assumed to be connected and players can communicate within their own team according to the network structure.

To learn a global NE, each team updates the state of its nodes using a distributed dual averaging method. The network-based information flow of our algorithm is related to [8], however, in a non-cooperative game-theoretic setup, convergence of such iterative methods to NE is non-trivial due to nature of equilibria, limited information exchange, etc. Sufficient conditions of convergence is further discussed in IV.

In our proposed algorithm, at each node  $i$  and iteration  $t$ , player  $\ell \in \{A, B\}$  maintains an estimate of its team's action as  $x_{\ell,i}(t)$ . A communication protocol is designed for sharing dual variables among the nodes of each team, where node  $i$  updates its dual variable  $z_{\ell,i}(t)$  using a convex combination of those of its neighboring teammates. Then it maps  $z_{\ell,i}(t)$  back to the set of admissible actions  $\mathcal{X}_\ell$  followed by taking its local action  $x_{\ell,i}(t)$ . Subsequently, players observe the actions of the opponent at each node and locally obtain an estimate of the subgradient of their distributed cost (reward). We show that under some regularity assumptions, this process provides players with enough local information to decide about the next step and eventually learn the global NE.

### C. Our Contribution: Team-based Dual Averaging

We assume that the structure of  $\mathcal{G}$  induces a doubly stochastic matrix  $P_\ell$  available to each team where  $P_{\ell,ij} > 0$

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### Algorithm 1 Team-Based Dual Averaging

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- 1: **Inputs:** For player  $\ell$
  - 2:     Subdifferential  $\partial f_{\ell,i}$  at node  $i$
  - 3:     Doubly stochastic  $P_\ell$  induced by network structure
  - 4: **Outputs:**
  - 5:     Estimates of NE  $\hat{x}_{\ell,i}(t)$  at node  $i$  for player  $\ell$
  - 6: **Initialize:**
  - 7:     Random:  $x_{\ell,0}(0), \dots, x_{\ell,n}(0)$
  - 8:      $t = 0, z_{\ell,i}(0) = 0$
  - 9: **while convergence**
  - 10:    **For** Player  $\ell = \{A, B\}$
  - 11:     **For** Node  $i = 1, \dots, n$
  - 12:        Update the dual variable  $z_{\ell,i}(t+1)$  by (4)
  - 13:        Calculate and take action  $x_{\ell,i}(t+1)$  by (4)
  - 14:     **For** Player  $\ell = \{A, B\}$
  - 15:        **For** Node  $i = 1, \dots, n$
  - 16:          Observe the opponents local action and get
  - 17:           $g_{\ell,i}(t+1) \in \partial_\ell f_{\ell,i}(x_{A,i}(t+1), x_{B,i}(t+1))$
  - 18:           $t = t + 1$
  - 19: **Return:** the *running average*  $\hat{x}_{\ell,i}$  by (6)
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if and only if nodes  $i$  and  $j$  are connected. Then each player  $\ell \in \{A, B\}$  at iteration  $t$  and each node  $i \in V$  computes updates by the following,

$$\begin{cases} z_{\ell,i}(t+1) &= \sum_{j \in \mathcal{N}_{\ell,i}} P_{\ell,ij} z_{\ell,j}(t) + \gamma_\ell(t) g_{\ell,i}(t) \\ x_{\ell,i}(t+1) &= \Pi_{\mathcal{X}_\ell}^\psi(-z_{\ell,i}(t+1), \alpha_\ell(t)), \end{cases} \quad (4)$$

where  $z_{\ell,i}$  and  $x_{\ell,i}$  are the dual variable and the local action of player  $\ell$  at node  $i$  respectively,  $g_{\ell,i}$  is an evaluation of subdifferential of the local cost  $f_{\ell,i}$  at the local actions  $x_{\ell,i}(t)$  as,

$$g_{\ell,i}(t) \in \partial_\ell f_{\ell,i}(x_{A,i}(t), x_{B,i}(t)), \quad (5)$$

where  $\partial_\ell$  is the differential w.r.t. the action of player  $\ell$ . Finally,  $\alpha_\ell$  and  $\gamma_\ell$  are positive step-sizes with  $\alpha_\ell$  being non-increasing. Note that  $x_{\ell,i}$  can be viewed as the local copy of  $x_\ell$  at node  $i$ , and its updates require access to only the  $i$ th row of the matrix  $P_\ell$ . We refer to the updates in (4) as *Team-Based dual averaging (TDA)*. The proposed methodology is summarized in Algorithm 1. We define the *running local average* at node  $i$ , for player  $\ell \in \{A, B\}$  as,

$$\hat{x}_{\ell,i}(t) = \frac{1}{t} \sum_{s=0}^t x_{\ell,i}(s). \quad (6)$$

In the sequel, we first show the convergence of function values,  $f_\ell$ , evaluated at  $\hat{x}_{\ell,i}$  to NE. Then, convergence of action iterates  $\hat{x}_{\ell,i}$  is discussed under more regularity conditions.

## IV. MAIN RESULTS

In this section, we present the main results of the paper and the corresponding analysis. We first make a few key assumptions that are used in our subsequent analysis.

**Assumption 1:** The undirected communication graph  $\mathcal{G}$  is connected.

**Assumption 2:** The cost functions  $f_A$  and  $f_B$  satisfy the following:

- 1) The cost  $f_A(x_1, x_2)$  is convex in  $x_1$  for any  $x_2$ , and concave in  $x_2$  for any  $x_1$ . Similarly, The cost  $f_B(x_1, x_2)$  is convex in  $x_2$  for any  $x_1$ , and concave in  $x_1$  for any  $x_2$ .
- 2) The cost  $f_A(x_1, x_2)$  is  $L_A$ -Lipschitz continuous in  $x_1$  for any  $x_2$  such that  $|f_A(x, y) - f_A(z, y)| \leq L_A \|x - z\|$ ,  $\forall x, z \in \mathcal{X}_A$ . Similarly, the cost  $f_B(x_1, x_2)$  is  $L_B$ -Lipschitz continuous in  $x_2$  for any  $x_1$ , i.e.,  $|f_B(x, y) - f_B(x, z)| \leq L_B \|y - z\|$ ,  $\forall y, z \in \mathcal{X}_B$ .

**Definition 1: (Cross-monotonicity)** A two-player game setup with cost functions  $f_1(x_1, x_2)$  and  $f_2(x_1, x_2)$  is *cross-monotone* if

$$\langle \mathcal{S}(x_1, x_2) - \mathcal{S}(x_1^*, x_2^*), (x_1, x_2) - (x_1^*, x_2^*) \rangle \geq 0, \quad (7)$$

for all  $(x_1, x_2), (x_1^*, x_2^*) \in \mathcal{X}_1 \times \mathcal{X}_2$  where  $\mathcal{S}$  is a 2-tuple defined as  $\mathcal{S}(x_1, x_2) = (\xi(x_1, x_2), \zeta(x_1, x_2))$  with any

$$\xi(x_1, x_2) \in \partial_1 f_2(x_1, x_2), \quad \zeta(x_1, x_2) \in \partial_2 f_1(x_1, x_2).$$

The above inner-product is defined over the product space  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ .

In contrast with optimization problems where the notion of ‘‘optimality’’ plays a central role, in a game setup, the objective is seeking an equilibrium rather than an ‘‘optimal’’ solution. In order to incorporate the interactions of players in convergence analysis of distributed algorithms, monotonicity has been known as a useful sufficient condition for problem ‘‘regularity,’’ originally due to the seminal work by Rosen [28]. The monotonicity as defined by Rosen, and subsequently used in the game literature (e.g., [29], [30] and references therein), refers to the property of the operator generated by the pseudo-gradient of the game. However, *cross-monotonicity* as proposed here is different and refers to the operator  $\mathcal{S}$  generated by the cross-pseudo-gradient as defined above. The reason for introducing this new regularity condition in the context of network games is due to the nature of our model, where the costs of players are distributed over the network rather than each node representing a player. The cross-monotonicity condition is used in the convergence of the function values in Section IV-C, while the conventional monotonicity is used to ensure the convergence of action iterates in Section IV-D.

#### A. Convergence Analysis

First, we introduce two lemmas that equip us with the required tools to prove the basic convergence. Herein, we borrow some tools from prior works, as such, we only mention the key steps that distinguishes our contribution in the game setup. Also, it is worth mentioning that even though we prove convergence of TDA with  $\gamma_\ell(t) = 1$ , it can improve the convergence rates numerically and its characteristics will be analyzed in a followup paper.

**Lemma 1:** Following Algorithm 1, suppose that player  $\ell \in \{A, B\}$  has access to  $g_{\ell, i}$  for  $(\ell, i) \in \mathcal{I}_\ell$  at each node  $i \in [n]$ , and  $\{\alpha_\ell(t)\}_{t=0}^\infty$  is a non-increasing sequence of

positive step-sizes and  $\gamma_\ell(t) = 1$ . Then under Assumption 2, for any  $x^* = (x_A^*, x_B^*) \in \mathcal{X}_A \times \mathcal{X}_B$  at node  $i$ ,

$$\begin{aligned} \mathcal{F}_\ell^{(i)}(T) &\leq \frac{1}{T} \frac{\psi(x_\ell^*)}{\alpha_\ell(T)} + \frac{1}{T} \frac{L_\ell^2}{2} \sum_{t=1}^T \alpha_\ell(t-1) \\ &\quad + \frac{L_\ell}{T} \sum_{t=1}^T \alpha_\ell(t) \left( \mathcal{R}_\ell^{(i)}(t) + \frac{2}{n} \sum_{j=1}^n \mathcal{R}_\ell^{(j)}(t) \right) \\ &\quad + \frac{1}{T} \frac{1}{n} \sum_{t=1}^T \sum_{j=1}^n \mathcal{H}_\ell^{(j)}(t), \end{aligned} \quad (8)$$

where,

$$\begin{aligned} \mathcal{F}_A^{(i)}(t) &= f_A(\hat{x}_{A,i}(t), x_B^*) - f_A(x^*), \\ \mathcal{F}_B^{(i)}(t) &= f_B(x_A^*, \hat{x}_{B,i}(t)) - f_B(x^*), \\ \mathcal{H}_A^{(i)}(t) &= \langle h_{A,i}^*(t) - h_{A,i}(t), x_{B,i}(t) - x_B^* \rangle, \\ \mathcal{H}_B^{(i)}(t) &= \langle h_{B,i}^*(t) - h_{B,i}(t), x_{A,i}(t) - x_A^* \rangle, \\ \mathcal{R}_\ell^{(i)}(t) &= \|\bar{z}_\ell(t) - z_{\ell,i}(t)\|_*, \end{aligned} \quad (9)$$

with  $\hat{x}_{\ell,i}(t)$  defined as in (6),  $\bar{z}_\ell(t) = \frac{1}{n} \sum_{i=1}^n z_{\ell,i}(t)$  as the averaging factor in the dual space, and finally  $h_{\ell,i}$  and  $h_{\ell,i}^*$  as the sub-gradients of  $f_{\ell,i}$  detailed below,

$$\begin{aligned} h_{A,i}(t) &\in \partial_B f_{A,i}(x_{A,i}(t), x_{B,i}(t)), & h_{A,i}^*(t) &\in \partial_B f_{A,i}(x^*), \\ h_{B,i}(t) &\in \partial_A f_{B,i}(x_{A,i}(t), x_{B,i}(t)), & h_{B,i}^*(t) &\in \partial_A f_{B,i}(x^*). \end{aligned} \quad (10)$$

*Proof:* Define  $y_\ell(t)$  for each team as the generalized projection,

$$\begin{aligned} y_\ell(t) &= \Pi_{\mathcal{X}_\ell}^\psi(-\bar{z}_\ell(t), \alpha) \\ &= \operatorname{argmin}_{x \in \mathcal{X}_\ell} \left\{ \sum_{s=1}^{t-1} \sum_{i=1}^n \langle g_{\ell,i}(s), x \rangle + \frac{1}{\alpha_\ell(s)} \psi(x) \right\}, \end{aligned} \quad (11)$$

which follows from the doubly stochastic nature of  $P_\ell$  and the iterative form of  $\bar{z}_\ell(t)$ ,

$$\bar{z}_\ell(t+1) = \bar{z}_\ell(t) + \sum_{j=1}^n g_{\ell,j}(t). \quad (12)$$

Convexity of  $f_A$  (in its first element) results in,

$$\mathcal{F}_A^{(i)}(t) \leq \frac{1}{T} \sum_{t=1}^T f_A(x_{A,i}(t), x_B^*) - f_A(x^*), \quad (13)$$

Using the  $L_A$ -Lipschitz property of  $f_A$  and  $\alpha$ -Lipchitz continuity of the generalized projection  $\Pi_{\mathcal{X}}^\psi(\cdot, \alpha)$  (see Lemma 1 in [3]) we can show that,

$$\begin{aligned} &\frac{1}{T} \sum_{t=1}^T f_A(x_{A,i}(t), x_B^*) - f_A(x^*) \\ &\leq \frac{1}{T} \sum_{t=1}^T f_A(y_A(t), x_B^*) - f_A(x^*) + \frac{L_A}{T} \sum_{t=1}^T \alpha_A(t) \mathcal{R}_A^{(i)}(t). \end{aligned} \quad (14)$$

By adding and subtracting  $f_{A,i}(x_{A,i}(t), x_B^*)$ , we can bound the first term on the right hand side of (14) as,

$$\begin{aligned} f_A(y_A(t), x_B^*) - f_A(x^*) &\leq \frac{L_A}{n} \sum_{i=1}^n \|y_A(t) - x_{A,i}(t)\| \\ &\quad + \frac{1}{n} \sum_{i=1}^n f_{A,i}(x_{A,i}(t), x_B^*) - f_{A,i}(x^*), \end{aligned} \quad (15)$$

where the  $L_A$ -Lipschitz condition is again leveraged. Then from the first part of Assumption 2 we can determine a bound on the last term in (15) as,

$$\begin{aligned} f_{A,i}(x_{A,i}(t), x_B^*) - f_{A,i}(x_A^*, x_B^*) &= \left( f_{A,i}(x_{A,i}(t), x_B^*) - f_{A,i}(x_{A,i}(t), x_{B,i}(t)) \right) \\ &\quad + \left( f_{A,i}(x_{A,i}(t), x_{B,i}(t)) - f_{A,i}(x_A^*, x_{B,i}(t)) \right) \\ &\quad + \left( f_{A,i}(x_A^*, x_{B,i}(t)) - f_{A,i}(x_A^*, x_B^*) \right) \\ &\leq \langle -h_{A,i}(t), x_{B,i}(t) - x_B^* \rangle + \langle g_{A,i}(t), x_{A,i}(t) - x_A^* \rangle \\ &\quad + \langle h_{A,i}^*(t), x_{B,i}(t) - x_B^* \rangle \\ &= \mathcal{J}_A^{(i)}(t) + \mathcal{H}_A^{(i)}(t), \end{aligned} \quad (16)$$

where,

$$\mathcal{J}_A^{(i)}(t) = \langle g_{A,i}(t), x_{A,i}(t) - x_A^* \rangle. \quad (17)$$

The rest of the proof contains bounding  $\mathcal{J}_A(t)$  using the  $\alpha$ -Lipchitz continuity of the generalized projection and the fact that  $\|g_{A,i}(t)\|_* \leq L_A$  which is skipped here due to brevity. The reader is referred to Lemma 3 in [8] for the details. Similar analysis results in the bound for  $\mathcal{F}_B^{(i)}(T)$ . ■

### B. Choice of the learning rate $\alpha_\ell(t)$

In order for Theorem 1 to result in convergence, an appropriate choice of step-size (learning rate)  $\alpha_\ell(t)$  is required. Next lemma shows how specific choices of  $\alpha_\ell(t)$  result in practical bounds on  $\mathcal{F}_\ell^{(i)}$  by getting rid of  $\mathcal{R}_\ell^{(k)}$  terms.

**Lemma 2:** Under Assumptions 1 and 2, and definitions of Lemma 1, suppose that  $\psi(x_\ell^*) \leq R_\ell^2$ . By choosing the step-size,

$$\alpha_\ell(t) = \frac{R_\ell \sqrt{1 - \sigma_2(P_\ell)}}{4\sqrt{2}L_\ell\sqrt{t}},$$

for  $\ell \in \{A, B\}$  and  $\gamma_\ell(t) = 1$ , at each node  $i$  we get,

$$\begin{aligned} \mathcal{F}_\ell^{(i)}(T) &\leq 8\sqrt{2} \frac{\log(T\sqrt{n})}{\sqrt{T}} \frac{R_\ell L_\ell}{\sqrt{1 - \sigma_2(P_\ell)}} \\ &\quad + \frac{1}{T} \frac{1}{n} \sum_{t=1}^T \sum_{j=1}^n \mathcal{H}_\ell^{(j)}(t). \end{aligned} \quad (18)$$

*Proof:* Stacking the updates of dual variables in (4) into a matrix form  $Z_\ell = [z_{\ell,1} \dots z_{\ell,n}]$  and similarly for  $G_\ell$ . Then for an undirected graph ( $P_\ell^\top = P_\ell$ ), we get

$$Z_\ell(t+1) = Z_\ell(t)P_\ell + G_\ell(t). \quad (19)$$

Define,

$$\Phi_\ell(t, s) = P_\ell^{t+1-s},$$

where  $\Phi$  can be a model of some transition matrix for a discrete-time linear system with states  $Z$ . This results in,

$$Z_\ell(t+1) = Z_\ell(s)\Phi_\ell(t, s) + \sum_{r=s+1}^t G_\ell(r-1)\Phi_\ell(t, r) + G_\ell(t)$$

for  $0 \leq s \leq t-1$ . Noting  $\Phi_\ell(t, s)\mathbb{1} = \mathbb{1}$  and from definition  $\bar{z}_\ell(t) = Z_\ell(t)\mathbb{1}/n$ , it is straightforward to show,

$$\begin{aligned} \bar{z}_\ell(t) - z_{\ell,i}(t) &= \bar{z}_\ell(s) - Z_\ell(s)\Phi_\ell(t, s)e_i \\ &\quad + \sum_{r=s+1}^{t-1} G_\ell(r-1)[\mathbb{1}/n - \Phi_\ell(t-1, r)e_i] \\ &\quad + G_\ell(t-1)[\mathbb{1}/n - e_i], \end{aligned}$$

where we used  $z_{\ell,i}(t) = Z_\ell(t)e_i$ . For simplicity we assume  $z_{\ell,i}(0) = \bar{z}_\ell(0)$  (say by choosing  $z_{\ell,i}(0) = 0$ ), then we have  $\bar{z}_\ell(s) - Z_\ell(s)\Phi_\ell(t, s)e_i = 0$  at  $s = 0$ . This implies,

$$\begin{aligned} \bar{z}_\ell(t) - z_{\ell,i}(t) &= \sum_{r=1}^{t-1} G_\ell(r-1)[\mathbb{1}/n - \Phi_\ell(t-1, r)e_i] \\ &\quad + G_\ell(t-1)[\mathbb{1}/n - e_i]. \end{aligned}$$

We can then proceed to bound this error,

$$\|\bar{z}_\ell(t) - z_{\ell,i}(t)\|_* \leq L_\ell \sum_{r=1}^{t-1} \|\mathbb{1}/n - \Phi_\ell(t-1, r)e_i\|_1 + 2L_\ell,$$

where we used  $\|g_{\ell,i}(t-1)\|_* \leq L_\ell$  and norm inequalities. Consider the following standard inequality ([31]),

$$\|\mathbb{1}/n - \Phi_\ell(t-1, r)e_i\|_1 \leq \frac{\sqrt{n}}{2} \sigma_2(P_\ell)^{t+1-r}.$$

Now we say  $r$  is small if  $t-r \geq \frac{\log(T\sqrt{n})}{\log \sigma_2(P_\ell)^{-1}} - 1$ , otherwise it is large. Then by splitting the sum, one would note that for small enough  $r$ ,

$$\|\mathbb{1}/n - \Phi_\ell(t-1, r)e_i\|_1 \leq \frac{1}{T},$$

and otherwise,

$$\|\mathbb{1}/n - \Phi_\ell(t-1, r)e_i\|_1 \leq 2.$$

Then it can be shown that,

$$\begin{aligned} \mathcal{R}_\ell^{(i)} &= \|\bar{z}_\ell(t) - z_{\ell,i}(t)\|_* \leq 2L_\ell \frac{\log(T\sqrt{n})}{\log \sigma_2(P_\ell)^{-1}} + 3L_\ell \\ &\leq 2L_\ell \frac{\log(T\sqrt{n})}{1 - \sigma_2(P_\ell)} + 3L_\ell, \end{aligned}$$

using  $\log \sigma_2(P_\ell)^{-1} \geq 1 - \sigma_2(P_\ell)$ . Then from Lemma 1,

$$\begin{aligned} \mathcal{F}_\ell^{(i)}(T) &\leq \frac{\psi(x_\ell^*)}{T\alpha_\ell(T)} + \frac{L_\ell^2}{2T} \sum_{t=1}^T \alpha_\ell(t-1) \\ &\quad + \frac{3L_\ell^2}{T} \left( 2 \frac{\log(T\sqrt{n})}{1 - \sigma_2(P_\ell)} + 3 \right) \sum_{t=1}^T \alpha_\ell(t) + \frac{1}{T} \frac{1}{n} \sum_{t=1}^T \sum_{j=1}^n \mathcal{H}_\ell^{(j)}(t). \end{aligned} \quad (20)$$

Define the sequence  $\{\alpha(t)\}_{t=0}^\infty$  as,

$$\alpha_\ell(t) = \frac{K_\ell}{\sqrt{t}}, \quad \alpha_\ell(0) = 1.$$

Since  $\psi(x_\ell^*) \leq R_\ell^2$  and  $\sum_{t=1}^T t^{-1/2} \leq 2\sqrt{T} - 1$ ,

$$\begin{aligned} \mathcal{F}_\ell^{(i)}(T) &\leq \frac{\sqrt{T}}{T} \left[ \frac{R_\ell^2}{K_\ell} + 19K_\ell L_\ell^2 + 12K_\ell L_\ell^2 \left( \frac{\log(T\sqrt{n})}{1 - \sigma_2(P_\ell)} \right) \right] \\ &\quad + \frac{1}{T} \frac{1}{n} \sum_{t=1}^T \sum_{j=1}^n \mathcal{H}_\ell^{(j)}(t). \end{aligned} \quad (21)$$

Choosing  $K_\ell = \frac{R_\ell \sqrt{1 - \sigma_2(P_\ell)}}{4\sqrt{2}L_\ell}$  results in (18).  $\blacksquare$

Now we are well-equipped to propose the main result of the paper.

### C. Convergence of Function Values

**Theorem 1:** Under Assumptions 1 and 2 and notations in Lemmas 1 and 2, if  $x^*$  is a global NE and the game is Cross-monotone, then for each node  $i$ ,

$$\lim_{T \rightarrow \infty} \mathcal{F}_A^{(i)}(T) = \lim_{T \rightarrow \infty} \mathcal{F}_B^{(i)}(T) = 0 \quad (22)$$

*Proof:* From Lemma 2,

$$\begin{aligned} \mathcal{F}_A^{(i)}(T) + \mathcal{F}_B^{(i)}(T) &\leq 8\sqrt{2} \frac{\log(T\sqrt{n})}{\sqrt{T}} \sum_{\ell \in \{A, B\}} \frac{R_\ell L_\ell}{\sqrt{1 - \sigma_2(P_\ell)}} \\ &\quad + \frac{1}{T} \frac{1}{n} \sum_{t=1}^T \sum_{j=1}^n \mathcal{H}_A^{(j)}(t) + \mathcal{H}_B^{(j)}(t). \end{aligned}$$

Expanding the the last term, and defining the operator  $S_j$  (as in (7) for  $f_1 = f_{A,j}$  and  $f_2 = f_{B,j}$ ) we get

$$\begin{aligned} \mathcal{H}_A^{(j)}(t) + \mathcal{H}_B^{(j)}(t) &= \\ &= -\langle S_j(x_{A,j}(t), x_{B,j}(t)) - S_j(x_A^*, x_B^*), (x_{A,j}(t), x_{B,j}(t)) - x^* \rangle \\ &\leq 0, \end{aligned}$$

since the game is cross-monotone at each node, i.e., the operator  $S_j$  is monotone, specifically at  $\xi = h_{B,j}$  and  $\zeta = h_{A,j}$ . Hence,

$$\mathcal{F}_A^{(i)}(T) + \mathcal{F}_B^{(i)}(T) \leq 8\sqrt{2} \frac{\log(T\sqrt{n})}{\sqrt{T}} \sum_{\ell \in \{A, B\}} \frac{R_\ell L_\ell}{\sqrt{1 - \sigma_2(P_\ell)}} \quad (23)$$

Then given that  $x^*$  is a global NE,  $\mathcal{F}_A^{(i)}(T), \mathcal{F}_B^{(i)}(T) \geq 0$  and the convergence will subsequently be guaranteed.  $\blacksquare$

### D. Convergence of Action Iterates

For simplicity, suppose that  $f_A$  and  $f_B$  are differentiable, then under Assumption 2.1 a NE exists (Theorem 1 in [28]). Define the operator  $\mathcal{Q}$  generated by the pseudo-gradient map as  $\mathcal{Q}(x_1, x_2) = (\omega(x_1, x_2), \nu(x_1, x_2))$  with

$$\omega(x_1, x_2) \in \partial_A f_A(x_1, x_2), \quad \nu(x_1, x_2) \in \partial_B f_B(x_1, x_2).$$

Additionally, suppose that  $\mathcal{Q}$  is (strictly) monotone, i.e.

$$\left\langle \mathcal{Q}(x_1, x_2) - \mathcal{Q}(x_1^*, x_2^*), (x_1, x_2) - (x_1^*, x_2^*) \right\rangle \geq 0, \quad (24)$$

for all  $(x_1, x_2), (x_1^*, x_2^*) \in \mathcal{X}_A \times \mathcal{X}_B$  with equality if and only if  $(x_1, x_2) = (x_1^*, x_2^*)$  (where inner-product is defined over the product space  $\mathbb{R}^{d_A} \times \mathbb{R}^{d_B}$ ). Then it follows that there exists a unique NE (see Theorem 2 in [28], where also a

sufficient condition for this monotonicity condition is given in terms of the game Jacobian).

Now by combining this uniqueness result with the convergence of TDA in function values, we conclude convergence of the running average of action iterates to the unique NE.

## V. EXAMPLE

In this section we illustrate the performance of our method for a two-team game on a network where each player has an objective,

$$f_{\ell,i} = \frac{1}{2} \|x_1 - a_{\ell,i}\|^2 + \frac{1}{4} \langle x_1, x_2 - b_{\ell,i} \rangle \quad (25)$$

where  $a_{\ell,i}$  and  $b_{\ell,i}$  are arbitrary prescribed parameters. The information about each player's cost is only known locally (at each node) and unknown to the opponent. Although the cost functions are locally Lipschitz, they can be treated as a Lipschitz continuous function over any (large) bounded domain. We have simulated the performance of TDA over complete, random 6-regular, and cycle graphs each consisting of 50 nodes. Figure 3(b) shows the random 6-regular network simulated here. To illustrate the advantage of dual averaging in our algorithm, we compare the results with another distributed algorithm — referred to as *Team-Based Mirror-Descent (TMD)* — which is an extension of *Distributed Mirror Descent* algorithm [7] implemented on our game model, where  $\psi(\cdot) = \frac{1}{2} \|\cdot\|^2$ . Also, as in standard form, the step-size of this algorithm  $\beta_\ell(t) = \frac{K_\ell}{t^{0.8}}$  is chosen to be not summable but square summable where  $K_\ell$  is the constant as in Lemma 2. The TMD algorithm is detailed in Algorithm 2.

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### Algorithm 2 Team-Based Mirror Descent

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- 1: **Inputs:** For player  $\ell \in \{A, B\}$
  - 2:   Subdifferential  $\partial f_{\ell,i}$  at node  $i$
  - 3:   Doubly stochastic  $P_\ell$  induced by network structure
  - 4: **Outputs:**
  - 5:   Estimates of NE as  $x_{\ell,i}(t)$  at node  $i$  for player  $\ell$
  - 6: **Initialize:**
  - 7:   Random:  $x_{\ell,0}(0), \dots, x_{\ell,n}(0)$
  - 8:    $t = 0$ ,
  - 9: **while convergence**
  - 10:   **For** Player  $\ell = \{A, B\}$
  - 11:     **For** Node  $i = 1, \dots, n$
  - 12:       Observe the opponents action:  $x_{-\ell,i}(t+1)$
  - 13:       Calculate the average estimate:
  - 14:        $v_{\ell,i}(t+1) = \sum_{j \in \mathcal{N}_{\ell,i}} P_{\ell,ij} x_{\ell,j}(t)$
  - 15:       Calculate get  $g_{\ell,i}(t+1) \in \partial_\ell f_{\ell,i}$  at  $v_{\ell,i}(t+1)$
  - 16:       and  $x_{-\ell,i}(t+1)$
  - 17:     **For** Player  $\ell = \{A, B\}$
  - 18:       **For** Node  $i = 1, \dots, n$
  - 19:         Calculate and take action
  - 20:          $x_{\ell,i}(t+1) = \text{Proj}_{\mathcal{X}_\ell} [v_{\ell,i}(t) - \beta_\ell(t) g_{\ell,i}(t)]$
  - 21:          $t = t + 1$
  - 22:     **Return**  $x_{\ell,i}(t)$  at node  $i$  for player  $\ell \in \{A, B\}$
-

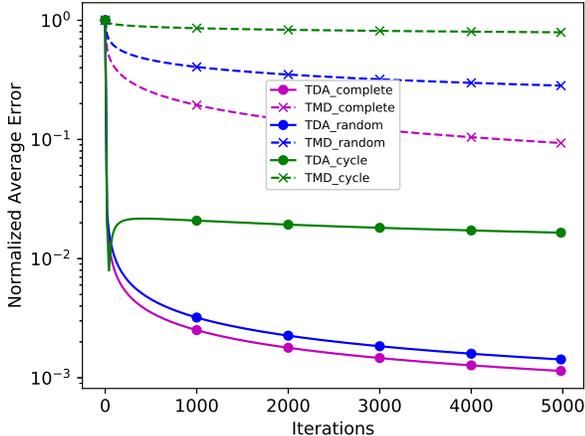


Fig. 2: NAE at each iteration for both TDA and TMD algorithms in complete, random 6-regular, and cycle graphs with 50 nodes.

We define Normalized Average Error (NAE) as the normalized mean of the running average error from NE over all nodes of the network as follows,

$$\text{NAE}(t) = \frac{\sum_{i=1}^n \|(\hat{x}_{A,i}(t) - x_A^*, \hat{x}_{B,i}(t) - x_B^*)\|}{\sum_{i=1}^n \|(\hat{x}_{A,i}(0) - x_A^*, \hat{x}_{B,i}(0) - x_B^*)\|}$$

Then, Figure 2 shows the NAE at each iteration for both of TDA and TMD algorithms. It can be noted that on the networks with the same structure, the TDA method has a convergence rate much faster than the TMD, which is due to dual averaging nature of TDA. Clearly, as the connectivity of the network decreases from complete to cycle, convergence rate becomes slower. This is captured in Theorem 1) where lower connectivity in the network leads to an increase of  $\sigma_2(P_\ell)$  closer to 1, which results in a larger bound for each time horizon  $T$ .

Next, we examine the performance of TDA in terms of the required iteration for a given error (from the NE). To do so, we consider four complete graphs with 25, 50, 80, and 120 nodes and require the algorithm to achieve a NAE of less than 0.1. Figure 3 illustrates the number of iterations  $T$  needed for each network to achieve  $\text{NAE}(T) < 0.1$ . Since, the initialization of the algorithm is random, for each network we have illustrated the mean and variance of the number of iterations required by 30 realizations. It is clear that, the number of required iterations to achieve the same error-bound, increases exponentially in this simulation. Part of this dynamic is due to the fact that TDA is a first-order method and its convergence in game setting is towards an equilibrium, unlike distributed optimization case where the convergence is towards an attractive optimal point.

## VI. CONCLUSION

In this paper, we proposed a new model for network games where each player's cost is distributed over a network. Without global information and only with distributed decision-making for players, we have shown that dual averaging

can be applied to the scenario where two teams, with distinct objectives, can coordinate with their respective team members, over the network. In particular, we have shown that using assumptions adopted in distributed optimization, in conjunction with a new cross-monotonicity condition on the game structure, the members of each team can distributively learn the NE.

Stochastic access to subdifferential of cost functions as well as noisy observations of opponent's action is considered as the next immediate extension of this work. Additionally, analysis of the step-size for the dual variables and equipping the algorithm with feasible second-order information are yet other achievable furtherance that can improve the convergence behavior of this method close to equilibrium.

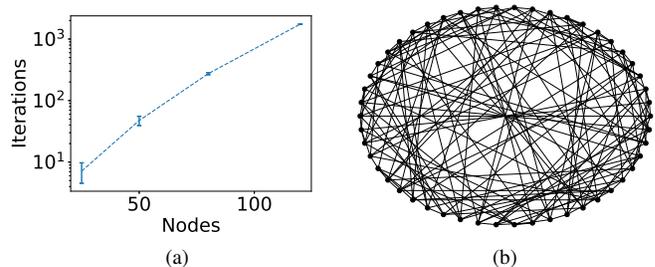


Fig. 3: (a) Number of iterations needed for each network so that NAE is less than 0.1, (b) The random 6-regular networks with 50 nodes simulated in our setup.

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