5.1

Quasigeostrophy

- Means the flow is close to geostrophic balance
- The "ageostrophic" part can be due to:
  - time dependence
  - advection
  - variation of f with latitude
  - friction (the "Ekman layer")
- This gives rise to important low frequency motions, that evolve in time even with $\omega \ll f$, notably "Rossby Waves."
- We do the derivation using SWE notation, but it is most important for baroclinic motions – so just use g' and H_{eff} instead of g + H when estimating things like wave speeds.
- decompose u + v into geostrophic + ageostrophic parts

$u = u_g + u_a$ | assume $[u_a, v_a] \ll [u_g, v_g]$

$v = v_g + v_a$ | define $u_g = -\frac{g}{f_0}\eta_y$, $v_g = \frac{g}{f_0}\eta_x$

Note $u_{gx} + v_{gy} = 0$

and $v_{gx} - u_{gy} \equiv \zeta_g = \frac{g}{f_0}(\eta_{xx} + \eta_{yy})$

Now, consider: $\boxed{x \text{ mom}}$ $\quad \dfrac{Du}{Dt} - fv = -g\eta_x \qquad f = f_0 + \beta y$

$$\sim \; u_{g_t} + \cancel{u_{a_t}} + \mathbf{u}_g\cdot\nabla u_g - \cancel{f_0 v_g} - f_0 v_a - \beta y v_g - \cancel{\beta y v_a} = -\cancel{g\eta_x}$$
$$+ \cancel{\mathbf{u}_g\cdot\nabla u_a}$$
$$+ \cancel{\mathbf{u}_a\cdot\nabla u_g}$$
$$+ \cancel{\mathbf{u}_a\cdot\nabla u_a}$$

/ means neglected because $[\mathbf{u}_a] \ll [\mathbf{u}_g]$

\\ means cancelled by geostrophy

dividing by f_0 + rearranging

$$\boxed{v_a = \frac{u_{gt}}{f_0} + \frac{\mathbf{u}_g\cdot\nabla u_g}{f_0} - \frac{\beta y v_g}{f_0}}$$

scaling $\quad \mathcal{U}_a = \mathcal{U}\dfrac{1}{f_0 T} \qquad \mathcal{U}\dfrac{\mathcal{U}}{f_0 L} \qquad \mathcal{U}\dfrac{\beta L}{f_0}$ $\qquad$ where $[\mathbf{u}_g] = \mathcal{U}$

$[\mathbf{u}_a] = \mathcal{U}_a$

to have $[\mathbf{u}_a] \ll [\mathbf{u}_g]$ then requires three things

$\dfrac{1}{f_0 T} \ll 1$ $\quad$ Temporal Rossby Number

$\dfrac{\mathcal{U}}{f_0 L} \ll 1$ $\quad$ Rossby Number

$\dfrac{\beta L}{f_0} \ll 1$ $\quad$ Fractional change of f

Note result for u_a: $\quad \boxed{u_a = -\dfrac{v_{gt}}{f_0} - \dfrac{\mathbf{u}_g\cdot\nabla v_g}{f_0} - \dfrac{\beta y u_g}{f_0}}$

For linear wave problems $\frac{U}{f_0 L} \ll \frac{1}{f_0 T}$ meaning parcels only move a small fraction of a wavelength as waves pass by. Then $\boxed{\text{mass}}$ (over a flat bottom) is

$$\eta_t + H(u_x + v_y) = 0$$

substitute in $\underset{\sim}{u} = \underset{\sim}{u}_g + \underset{\sim}{u}_a$, noting:

$$u_a = -\frac{g}{f_0^2}\eta_{xt} + \frac{\beta y g}{f_0^2}\eta_y$$

$$v_a = -\frac{g}{f_0^2}\eta_{yt} - \frac{\beta y g}{f_0^2}\eta_x$$

$$\Rightarrow \eta_t + H(\underbrace{\cancel{u_{gx} + v_{gy}}}_{0}) + H(\underbrace{u_{ax} + v_{ay}}) = 0$$

$$-\frac{g}{f_0^2}(\eta_{xx} + \eta_{yy})_t + \frac{\beta y g}{f_0^2}(\underbrace{\cancel{\eta_{yx} - \eta_{xy}}}_{0}) - \frac{\beta g}{f_0^2}\eta_x$$

$$\Rightarrow \boxed{\left[(\eta_{xx} + \eta_{yy}) - \frac{1}{a^2}\eta\right]_t + \beta\eta_x = 0}$$

= The linear, SWE "Quasigeostrophic Potential Vorticity" (QGPV) Eqn.

Note $a = \frac{\sqrt{gH}}{f_0}$

~ rate of change of ζ ~ stretching ~ change of latitude

We could also have derived this from $\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$ under the same assumptions.