IGW: Group Velocity & Flow over Topography

Recall: IGW dispersion relation:
\[ \omega = \frac{f^2 m^2 + N^2 k^2}{m^2 + k^2} \quad \text{(\because)} \]

For hydrostatic flow, we drop this term:
\[ \nabla^2 \omega = \frac{\partial}{\partial t} \left[ -\frac{f}{c_0} \frac{\partial \rho}{\partial t} + B \right] \quad \text{in derivation. neglect.} \]

\( \tan \alpha = \frac{k}{m} \quad \text{and} \quad \alpha = \frac{\omega^2 - f^2}{N^2 - \omega^2} \quad \text{from (\*)} \)

Stll 8.1 defines 3 regimes:

1. \( \omega \) close to \( N \) Non-hydrostatic, Non-rotating \( \alpha \sim 1 \)

2. \( f \ll \omega \ll N \) Hydrostatic, Non-rotating \( \alpha \sim 0.1 \)

3. \( \omega \) close to \( f \) Hydrostatic, Rotating \( \alpha \sim 0.01 \)

Energy - like wave packets travels with the group velocity.

\[ C_g = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right) \]

and from (\*) we can show \( C_g \perp k \)

\( C_g \) is normal to phase surface.

\[ \nabla \times \omega \quad \text{wave packet} \quad \nabla \cdot C_p \quad \text{A upward phase propagation has downward energy propagation!} \]
Consider IGW's forced by flow over sinusoidal hills:
- a steady flow
- \( \frac{\partial \mathbf{v}}{\partial y} = 0 \) (except \( \mathbf{f} \mathbf{v} = -\frac{1}{\rho} \mathbf{g} \nabla h \))
- \( \mathbf{v} = (u, v, w) \)

Wind: \(-2U\) (\(U = \text{positive, const.}\))

We could derive a solution with \( \mathbf{u}' = (-U + u', v', w') \)

then \( \frac{D}{Dt} = -U \frac{\partial}{\partial x} \) instead of \( \frac{\partial}{\partial t} \)

Or make a Galilean transformation into four moving

at velocity \(-U\) (to the left, with speed \(U\))

\( \text{Now fluid is motionless except for waves, and the ground is moving} \)

\[ z_b = \delta \sin \left[ k(x - Ut) \right] \]
this gives a b.c. for the wave problem

\[ w (z = z_b) \approx w (z = 0) = \frac{\partial \tilde{b}}{\partial t} = -\delta k U \cos \left[ k (x - z U t) \right] \]

The general solution is \[ w = \text{Re} \{ w_0 \exp(i(kx + mz - \omega t)) \} \]

\[ w = -\delta k U \cos \left( kx + mz - \omega t \right) \]

with:
- \( k \) set by topography
- \( \omega = \frac{U}{k} \) set by frequency of parcel encountering with hills

Still need to know \( m \)...

For simplicity, assume flow is in regime \( \Theta \): Hydrostatic, non-rotating

\[ \omega^2 = \frac{k^2 m^2 + N^2 k^2}{m^2 + k^2} \Rightarrow \omega = \pm \frac{Nk}{m} \text{ (ii)} \]

\[ \Rightarrow \frac{C_q^2}{2m} = \frac{\partial \omega}{\partial m} = \mp \frac{Nk}{m^2} \text{ (ii)}, \] physically, we require

upward energy propagation (forced from below)

so, assuming \( k \) positive, we take positive root in (ii)

hence negative root in (i) \[ \Rightarrow \frac{m}{\omega} = \frac{-Nk}{\omega} = -\frac{N}{U} \text{ (negative)} \]
Solution sketch

\[ \phi = 0 \quad \phi = 2\pi \]

\[ \begin{align*}
\alpha^2 &= \frac{w^2 - \alpha^2}{N^2 - \alpha^2} = \frac{w^2}{N^2} \\
\Rightarrow \quad \alpha &= \pm \frac{w}{N} = \frac{4h}{N} \left( \frac{U}{h} \right) = -\frac{7h}{N} \frac{U}{h}
\end{align*} \]

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\[ \frac{2h}{N} \text{ smaller} \quad \frac{2h}{N} \text{ bigger} \]

A pattern for real mountain waves is more complex.

\[ \text{Lee Waves} \quad \text{Lenticular Cloud} \quad \text{Wind} \]

\[ \text{Rotor} \quad \text{"Form drag"} \]