4.4 Internal Gravity Waves : IGW's

Assume Boussinesq, f-plane flow, but with continuous stratification.

$\Rightarrow$ vertically-propagating internal waves, or higher vertical mode numbers.

(Note: we will allow non-hydrostatic solutions)

Consider a fluid with $\rho = \rho_0 + \bar{\rho}(z) + \rho'(\underset{\sim}{x}, t)$ } potential density

and $[\rho'] \ll [\bar{\rho}] \ll \rho_0$ so these are small perturbations away from a mean stratification

(e.g. atm. internal waves with vertical motions small compared to the scale height.)

Scaling of $\boxed{\text{mass}}$ $\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \underset{\sim}{u} = 0$

Flow is Boussinesq, so $\nabla \cdot \underset{\sim}{u} = 0 \Rightarrow W = U\frac{H}{L}$

But it is also stratified so we make use of $\frac{D\rho}{Dt} = 0$

$$\Rightarrow \rho'_t + \overset{①}{w\bar{\rho}_z} + \overset{②}{\underset{\sim}{u} \cdot \nabla\rho'} = 0$$

scales $\quad \frac{U\cancel{H}}{L}\frac{[\bar{\rho}]}{\cancel{H}} \quad \frac{U[\rho']}{L} \quad \Rightarrow ② \ll ①$

because $[\rho'] \ll [\bar{\rho}]$

so $\boxed{\rho'_t + w\bar{\rho}_z = 0}$ call this $\boxed{\rho}$

Means: variation of ρ' is due primarily to vertical advection of the background stratification.

We decompose the pressure as $p = \bar{p}(z) + p'(\underset{\sim}{x}, t)$

and <u>define</u> $\bar{p}$ by: $\bar{p}_z = -(\rho_0 + \bar{\rho})g$ (*)

The linear, Boussinesq, f-plane momentum equations are

$\boxed{\text{x mom}}$ $\boxed{u_t - fv = -\frac{1}{\rho_0} p'_x}$ Note: $\bar{p}_x = \bar{p}_y = 0$

$\boxed{\text{y mom}}$ $\boxed{v_t + fu = -\frac{1}{\rho_0} p'_y}$

cancel by (*)

$\boxed{\text{z mom}}$ $w_t = -\frac{1}{\rho_0}\cancel{\bar{p}_z} - \frac{1}{\rho_0} p'_z - \cancel{\frac{g(\rho_0 + \bar{\rho})}{\rho_0}} - \frac{g\rho'}{\rho_0}$

$\Rightarrow$ $\boxed{\text{z mom}}$ $\boxed{w_t = -\frac{1}{\rho_0} p'_z - \frac{g\rho'}{\rho_0}}$ ← can be hydrostatic or non-hydrostatic

and $\boxed{\text{mass}}$ $\boxed{\nabla \cdot \underset{\sim}{u} = 0}$

$\boxed{\rho}$ $\boxed{\rho'_t + w\bar{\rho}_z = 0}$

5 equations in u, v, w, p', ρ' ✓

The procedure to form a single PDE in w is similar to how we got Poincaré waves.
[see Gill 8.4 for details]

- One step is that we take the curl: $\boxed{y \text{ mom}}_x - \boxed{x \text{ mom}}_y$
which gives $\boxed{q_t = f w_z}$ which is an expression of PV conservation for continuous stratification
[we used the property from $\boxed{\text{mass}}$ $w_z = -(u_x + v_y)$]

The end result is one equation in w: (see Appendix!)

$$\boxed{(\nabla^2 w)_{tt} + f^2 w_{zz} + N^2 (w_{xx} + w_{yy}) = 0} \quad (*)$$

$$N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}}$$

for wave solutions of the form

$$w = \text{Re}\left\{ w_0 \exp i(kx + ly + mz - \omega t)\right\}$$

vertical wave number "m"

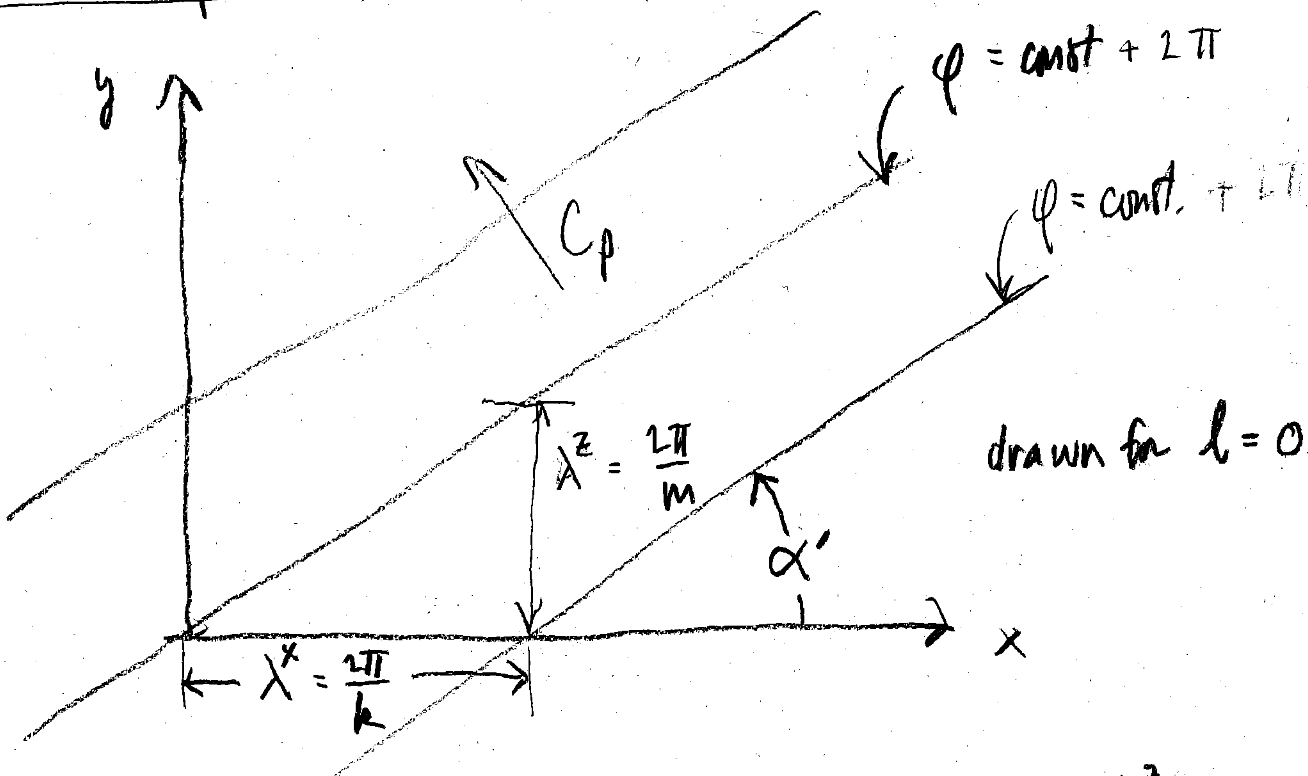
plugging this into (*) gives the dispersion relation:

$$\boxed{\omega^2 = \frac{f^2 m^2 + N^2 K_H^2}{m^2 + K_H^2}} \quad (**) \quad \text{where } K_H^2 \equiv k^2 + l^2$$

Solution Properties:

phase: $\varphi \equiv kx + ly + mz - \omega t$

surfaces of constant φ ("wave crests") are inclined planes in 3-D space



drawn for $l = 0$

"aspect ratio" $\alpha = \frac{k}{m}$ (generally $\frac{K_H}{m}$) $= \frac{\lambda^z}{\lambda^{x,y}} = \tan\alpha'$

from (**) we may show $\alpha^2 = \frac{\omega^2 - f^2}{N^2 - \omega^2}$

limits: ① $m^2 \ll K_H^2 \Rightarrow \omega \to N$ φ-surfaces ~ vertical |||

② $K_H^2 \ll m^2 \Rightarrow \omega \to f$ φ-surfaces ~ flat $\equiv$

Wave solutions have ω between f + N ! ☆☆

Appendix: details of the IGW equation development

Boussinesq, $f = \text{const.}$

continuous stratification

$$\begin{aligned}
&\boxed{x\ \text{mom}} \quad u_t - fv = -\frac{1}{\rho_0} p'_x \\
&\boxed{y\ \text{mom}} \quad v_t + fu = -\frac{1}{\rho_0} p'_y \\
&\boxed{z\ \text{mom}} \quad w_t = -\frac{1}{\rho_0} p'_z + b
\end{aligned}$$

where $b \equiv -g\rho'/\rho_0$ "buoyancy"

$$\boxed{\rho} \quad \frac{g}{\rho_0}\left[\rho'_t + w\bar{\rho}_z = 0\right]$$

$$\begin{aligned}
\Rightarrow\ &\boxed{\rho} \quad b_t + wN^2 = 0 \\
&\boxed{\text{mass}} \quad u_x + v_y + w_z = 0
\end{aligned}$$

where $N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}}$ "buoyancy frequency" (assume constant)

Take curl: $\boxed{y\ \text{mom}}_x + \boxed{x\ \text{mom}}_y$

$$\Rightarrow \zeta_t + f(u_x + v_y) = 0$$

where $\zeta \equiv v_x - u_y$ "relative vorticity"

or using $\boxed{\text{mass}}$ $\Rightarrow$ $\boxed{\zeta_t = f w_z}$ (*) PV conservation

so vertical stretching causes changes in ζ

Then forming

$$\frac{\partial}{\partial t}\boxed{z\ \text{mom}} \Rightarrow \quad w_{tt} = -\frac{1}{\rho_0} p'_{zt} + b_t$$

substitute in from $\boxed{\rho}$ $\quad b_t = -w N^2$

and take $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ of the result

$$\Rightarrow \boxed{(w_{xx} + w_{yy})_{tt} = -\frac{1}{\rho_0}\left(p'_{xxzt} + p'_{yyzt}\right) - (w_{xx} + w_{yy})N^2} \quad (**)$$

We are working toward a single equation in w, so let's try to get rid of the p' terms in equation $(**)$. To do this, form the horizontal divergence equation $\boxed{x\ \text{mom}}_x + \boxed{y\ \text{mom}}_y$

$$\Rightarrow \quad (u_x + v_y)_t - f(v_x - u_y) = \frac{-1}{\rho_0}(p'_{xx} + p'_{yy})$$

or $\quad -w_{zt} - f\zeta = -\frac{1}{\rho_0}(p'_{xx} + p'_{yy})$ $\quad$ and take $\frac{\partial}{\partial t}$

$$\Rightarrow \quad -w_{ztt} - f\zeta_t = -\frac{1}{\rho_0}(p'_{xx} + p'_{yy})_t$$

then use $(*)$ to rewrite the ζ term, giving $\;-w_{ztt} - f^2 w_z = \frac{-1}{\rho_0}\left(p'_{xx} + p'_{yy}\right)_t$

and take $\frac{\partial}{\partial z}$

$$\Rightarrow \quad -w_{zztt} - f^2 w_{zz} = -\frac{1}{\rho_0}\left(p'_{xxzt} + p'_{yyzt}\right)$$

then substituting this into (**) we get the desired result

$$(w_{xx} + w_{yy})_{tt} + w_{zztt} + f^2 w_{zz} + N^2(w_{xx} + w_{yy}) = 0$$

or

$$\boxed{(\nabla^2 w)_{tt} + f^2 w_{zz} + N^2(w_{xx} + w_{yy}) = 0}$$

which is the result from page 3 of this lecture.

There is a nice discussion in Holton 7.4 of the derivation expressed in atmospheric terms such as potential temperature. The derivation here closely follows Gill 8.4, and expressions for the other parts of the solution, such as u, v and ρ' are found in Gill 8.5.