$\boxed{4.2}$ Take differences of the two layer equs.

• define $\Delta u = u_1 - u_2$, $\Delta v = v_1 - v_2$

$$\Rightarrow \Delta u_t - f \Delta v = g' E_x$$ eq. from $\boxed{X\ mom}_1 - \boxed{x\ mom}_2$

$$\Delta v_t + f \Delta u = g' E_y$$

and forming $H_2 \boxed{mass}_1 - H_1 \boxed{mass}_2$

$$\Rightarrow H_2\left(\frac{1}{\mu} - 1\right) E_t - H_1 E_t + H_1 H_2 (\Delta u_x + \Delta v_y) = 0$$

recall $E = \mu\eta$ + $\eta = \frac{1}{\mu} E$

the mode with big internal displacements has $[E] \gg [\eta]$

so $[\mu] \gg 1$ and $\frac{1}{[\mu]} \ll 1$

(called the "rigid lid approximation")

$$\Rightarrow \boxed{mass} \text{ becomes } -E_t + \frac{H_1 H_2}{H_1 + H_2}(\Delta u_x + \Delta v_y) = 0$$

and $\boxed{x\ mom}$ $\Delta u_t - f \Delta v = -g'(-E)_x$

$\boxed{y\ mom}$ $\Delta v_t + f \Delta u = -g'(-E)_y$

These are mathematically identical in form to the Poincaré Wave solution, with

Poincaré	2-Layer
η $\longrightarrow$	$-E$
u $\longrightarrow$	Δu
v $\longrightarrow$	Δv
H $\longrightarrow$	$H_1 H_2/(H_1+H_2) \equiv H_{eff}$ the "effective depth"
g $\longrightarrow$	g'

Thus the solutions will satisfy

$$E_{tt} + f^2 E - g' H_{eff} (E_{xx} + E_{yy}) = 0$$

$\Rightarrow$ dispersion relation $\omega = (K^2 c^2 + f^2)^{\frac{1}{2}}$, $K = k^2 + l^2$

and $C = \sqrt{g' H_{eff}}$

This is the baroclinic mode: zero net transport

so eg. $H_1 u_1 + H_2 u_2 = 0$, so $H_1 \boxed{x\ mom}_1 + H_2 \boxed{x\ mom}_2$:

$$\Rightarrow \quad 0 = -g(H_1 + H_2)\eta_x - g' H_2 E_x$$

or, using $E = \mu\eta$

$$0 = \left[-g(H_1 + H_2) - g' H_2 \mu\right] \eta_x$$

start here.

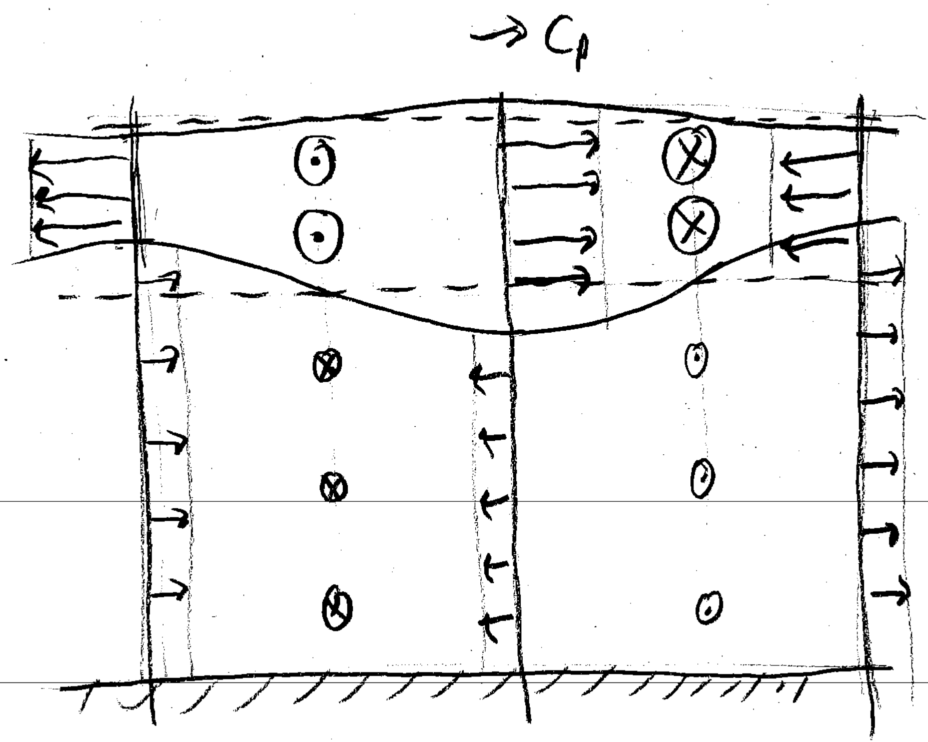
for this to be true for all η_x we require

$$\mu = -\frac{g(H_1 + H_2)}{g' H_2}$$

: - large in magnitude
- negative in sign

} consistent with our earlier assumptions ✓

Solution sketch:



- PV is conserved in each layer
- u reverses w/depth
- u can be much bigger now because E is big.

The other, "barotropic," mode has $[\mu] \sim \mathcal{O}(1)$ and is almost the same as 1 layer (unstratified SWE flow.

Barotropic mode	Baroclinic mode
~ 1 layer	2 layer
max wave speed $= \sqrt{gH}$	$= \sqrt{g' H_{eff}}$ slow!
no vertical shear	big vertical shear $(\Delta u, \Delta v)$
Interface follows free surface w/ smaller displacement $E = \left(\frac{H_2}{H_1 + H_2}\right)\eta$	Interface down when surface is up, and it moves a lot $E = -\frac{g}{g'}\left(\frac{H_1 + H_2}{H_2}\right)\eta$

e.g. ocean thermocline $\quad g' = \frac{g\Delta\rho}{\rho_0} = \frac{10 \times 2}{10^3} \frac{m}{s^2} = 2 \times 10^{-2}\ m\,s^{-2}$

$$H_{eff} = \frac{500\,m \cdot 3500\,m}{4000\,m} = 437\ m$$

$\Rightarrow\ c = \sqrt{g' H_{eff}} = 3\ m\,s^{-1}$, vs. $\sqrt{gH} = 200\ m\,s^{-1}$

for atm weather case: $g' = 1\ m\,s^{-2}$, $H_{eff} = 2.5\ km$, $c = \sqrt{g' H_{eff}} = 50\ m\,s^{-1}$