4.1 Shallow Water Equations with 2-Layer Stratification

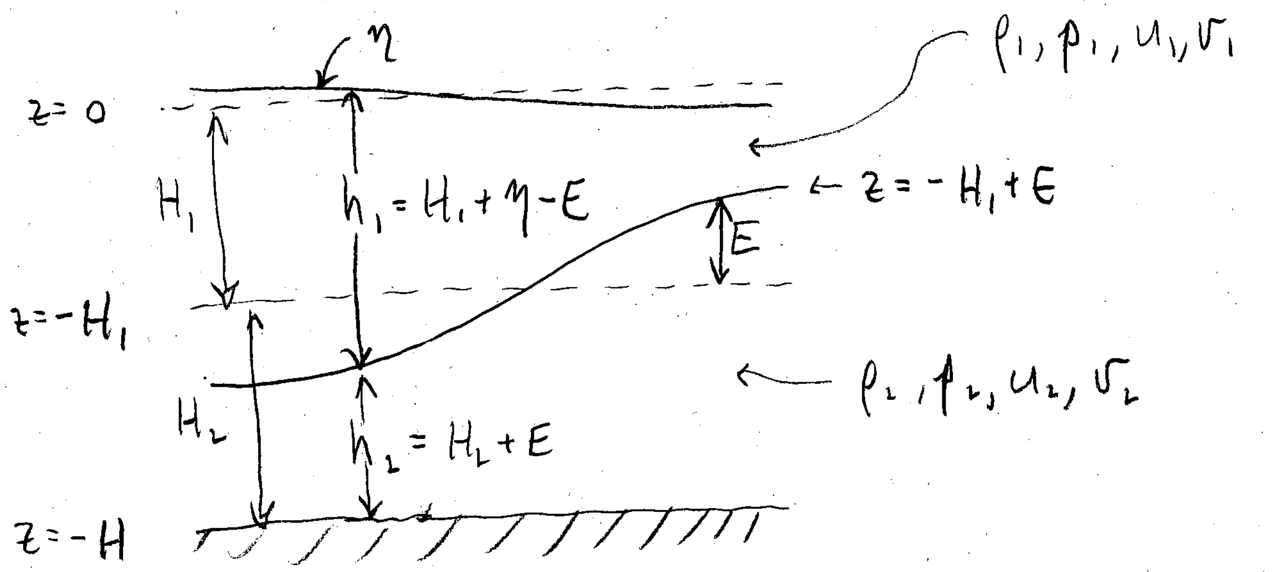
Most important atm. + ocean flows are <u>stratified</u>

meaning $\frac{\partial \rho_{pot}}{\partial z} < 0$, which has two consequences

(i) they develop vertical shear $\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \neq 0$

(ii) disturbances of potential density surfaces (weather systems, eddies, waves) evolve much more slowly than for 1 layer.

(iii) these properties are mostly captured with just two layers:



$= -(H_1 + H_2)$

Physically these are layers of constant <u>potential</u> density (~const. potential temperature Θ)

Note: fluid is "barotropic" within each layer, $\rho = \rho(p)$

[see Holton 3.4]

Typical Values

	H_1 / H_2		ρ_0 (all ρ_{pot})	$\Delta\rho = \rho_2 - \rho_1$ (all ρ_{pot})
Weather Systems	5 km / 5 km	$\theta_1 \sim 320\,K$, $\theta_2 \sim 290\,K$	$0.7\ kg\,m^{-3}$	$.07\ kg\,m^{-3}$ (10%)
Atm. with low level inversion or coastal marine boundary layer	9 km / 1 km	$\Delta\theta = 10\text{-}20\,K$	$1.2\ kg\,m^{-3}$	$.06\ kg\,m^{-3}$ (5%)
Gulf Stream	800-400 m / 4 km	$\theta_1 = 17°C$, $\theta_2 = 5°C$	$1025\ kg\,m^{-3}$	$2\ kg\,m^{-3}$ (0.2%)

- We will consider linear, hydrostatic motion, f-plane

Assume Boussinesq approx. is valid $[\rho'] \ll \rho_0 \Rightarrow \nabla \cdot \underline{u} \cong 0$

so we ignore compressibility - except that we use scales from ρ_{pot} (also see Vallis 3.9)

$$\Rightarrow \quad p_1 = p_{above} + \rho_1 g(\eta - z)$$

$$p_2 = p_{above} + \rho_1 g h_1 + \rho_2 g(-H_1 + E - z)$$

where p_{above} assumed constant (eg. p_{atm} for the ocean)

$$\Rightarrow \quad p_{1x} = \rho_1 g \eta_x$$

$$p_{2x} = \rho_1 g \underbrace{(\eta_x - E_x)}_{\text{from } h_{1x}} + \rho_2 g E_x = \rho_1 g \eta_x + \Delta\rho\, g E_x$$

- and the momentum equations are

$$u_{1t} - f v_1 = -g\eta_x$$

$$v_{1t} + f u_1 = -g\eta_y$$

$$u_{2t} - f v_2 = -g\eta_x - g' E_x$$

$$v_{2t} + f u_2 = \underbrace{-g\eta_y}_{\text{approximating } \rho_2/\rho_1 \simeq 1 \text{ (Boussinesq)}} - g' E_y$$

define

$$g' = \frac{g\Delta\rho}{\rho_1}$$

the "reduced gravity"

start here

- Mass conservation:

$$h_{1t} + (h_1 u_1)_x + (h_1 v_1)_y = 0 \qquad \boxed{\text{mass}}_1$$

$$h_{2t} + (h_2 u_2)_x + (h_2 v_2)_y = 0 \qquad \boxed{\text{mass}}_2$$

Linearize assuming $[\eta] + [E] \ll H_1 + H_2$

$$\Rightarrow \quad \eta_t - E_t + H_1(u_{1x} + v_{1y}) = 0$$

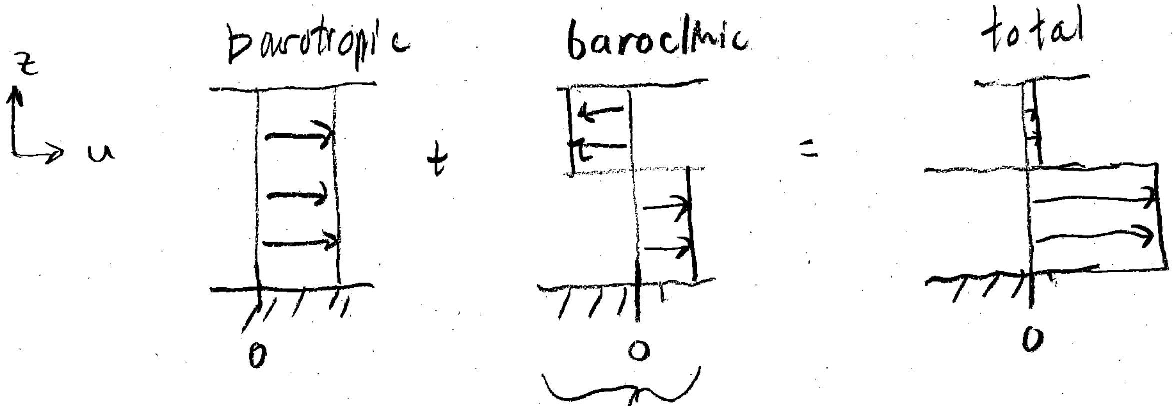
$$E_t + H_2(u_{2x} + v_{2y}) = 0$$

To find solutions we take $\frac{\partial}{\partial t}\boxed{\text{mass}}$ and

$\boxed{y\text{ mom}}_x - \boxed{x\text{ mom}}_y$ as for Poincaré waves

$\Rightarrow$ single 4^{th} order equation in η (ack!)

- **But** we can simplify by representing the velocity field as the sum of two "modes"



defined as having zero vertically-integrated transport

For wavelike solutions for either mode the free surface + interface must be linked, so we look for solutions with $E = \mu \eta$ (so $\eta = E/\mu$)

where μ is to be determined for each mode.

to be continued ...