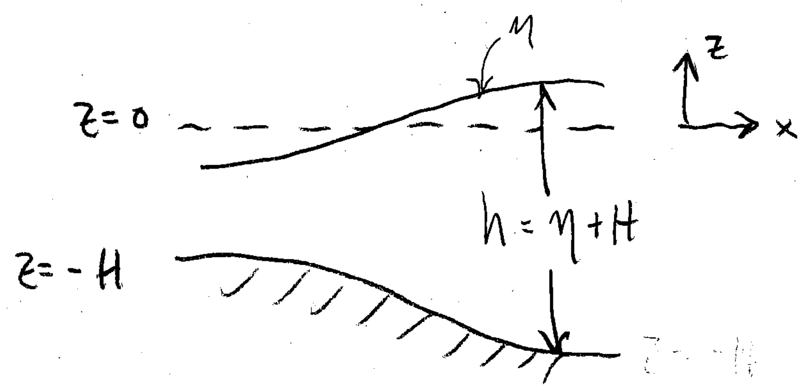
3.6 SWE Potential Vorticity Conservation with Topography ; Taylor Columns

- Retain nonlinear terms
- still hydrostatic
- $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$
- $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$
- allow $f(y)$

$\boxed{x \text{ mom}}$ $\quad \frac{Du}{Dt} - fv = -g\eta_x$

$\boxed{y \text{ mom}}$ $\quad \frac{Dv}{Dt} + fu = -g\eta_y$

$\boxed{\text{mass}}$ $\quad \eta_t + (hu)_x + (hv)_y$



- Rewrite $\boxed{\text{mass}}$ as

$$\underbrace{\underbrace{\eta_t + \overset{=0}{H_t}}_{h_t} + \overbrace{uh_x + vh_y + h(u_x + v_y)}^{(hu)_x + (hv)_y}}_{\equiv Dh/Dt} = 0$$

$$\Rightarrow \quad \boxed{\frac{Dh}{Dt} + h(u_x + v_y) = 0}$$

- Take $\boxed{y \text{ mom}}_x - \boxed{x \text{ mom}}_y$

$$\Rightarrow \quad \underbrace{\frac{\partial}{\partial x}\frac{Dv}{Dt} - \frac{\partial}{\partial y}\frac{Du}{Dt}}_{\text{I}} + f(u_x + v_y) + \underbrace{v\frac{df}{dy}}_{= \frac{Df}{Dt}} = 0 \qquad (*)$$

rewrite $\underline{I}$ (note $\zeta = v_x - u_y$: vorticity) ②

$$\underline{I} = \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left(u v_x + v v_y\right) - \frac{\partial}{\partial y}\left(u u_x + v u_y\right)$$

$$= \zeta_t + u_x v_x + v_x v_y - u_y u_x - v_y u_y$$

$$+ u \frac{\partial}{\partial x}(v_x - u_y) + v \frac{\partial}{\partial y}(v_x - u_y)$$

$$= \zeta_t + u\zeta_x + v\zeta_y + (u_x + v_y)\zeta$$

$$= \frac{D\zeta}{Dt} + (u_x + v_y)\zeta \quad = \underline{I}$$

$\Rightarrow$ (*) is $\dfrac{D\zeta}{Dt} + (\zeta + f)(u_x + v_y) + \dfrac{Df}{Dt} = 0$

or $\boxed{\dfrac{D}{Dt}(\zeta + f) + (\zeta + f)(u_x + v_y) = 0}$

and from $\boxed{\text{mass}}$ $-\dfrac{1}{h}\dfrac{Dh}{Dt} = (u_x + v_y)$

$$\Rightarrow \frac{D}{Dt}(\zeta + f) - \left(\frac{\zeta + f}{h}\right)\frac{Dh}{Dt} = 0$$

or $\dfrac{1}{h}\dfrac{D}{Dt}(\zeta + f) - \dfrac{\zeta + f}{h^2}\dfrac{Dh}{Dt} = 0$ & note $\dfrac{D}{Dt}\left(\dfrac{1}{h}\right) = -\dfrac{1}{h^2}\dfrac{Dh}{Dt}$

so we may write

$$(+) \quad \boxed{\frac{DQ}{Dt} = 0} \qquad \text{where} \quad Q \equiv \frac{\zeta + f}{h} = \text{potential vorticity}$$

"PV"

$\Rightarrow$ PV is conserved following fluid columns

Note: for $H = \text{const}$, $\eta \ll H$, $\zeta \ll f$ this reduces to our old result $\left(\zeta - \frac{f}{H}\eta\right)_t = 0$

Note: scale of $[\zeta] = \frac{U}{L}$ so $\left[\frac{\zeta}{f}\right] = \frac{U}{fL} = R_o \quad (++)$

Previously the Rossby # was the scale of

$$\left[\frac{D\underline{u}_H/Dt}{f\hat{k} \times \underline{u}}\right] \text{ in } \boxed{x, y \text{ mom}}$$

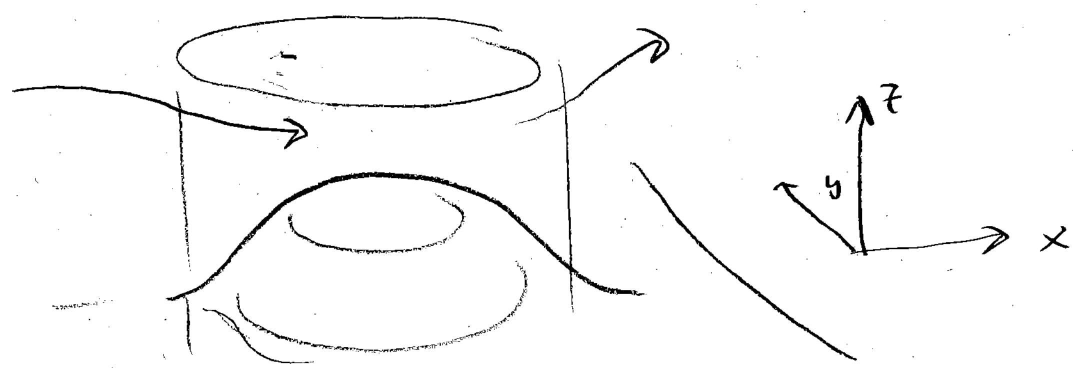
so $(++)$ is another interpretation

for $R_o \ll 1 \Rightarrow \zeta \ll f$

and $Q = \dfrac{f}{h}$ is conserved, to $O(R_o)$, by fluid columns.

for variations of H >> variations of η

fluid follows topographic contours



Tend to get closed streamlines over a bump

called "Taylor Column"