3.3

Properties of Poincaré Waves, Inertial Oscillations

Shallow Water Equations, f-plane, linearized

$\boxed{x \text{ mom}}$ $u_t - fv = -g\eta_x$

$\boxed{y \text{ mom}}$ $v_t + fu = -g\eta_y$

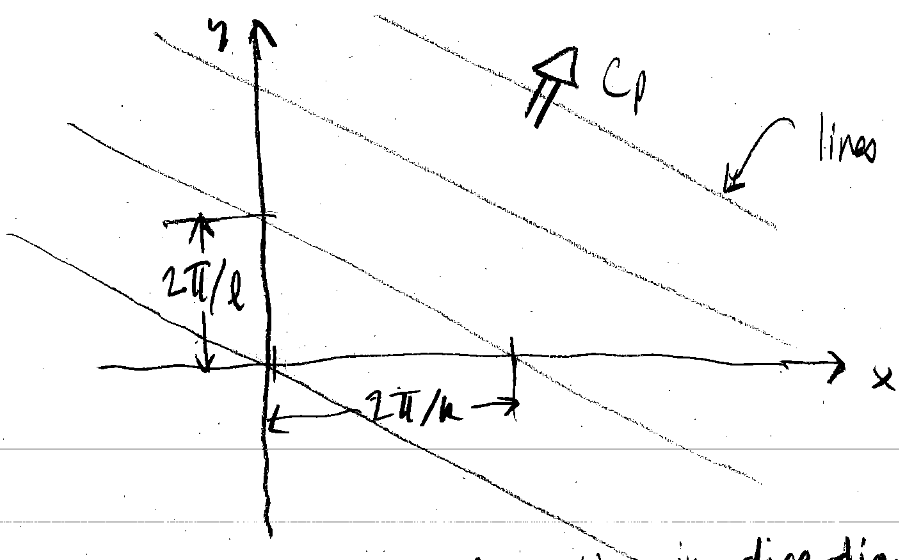
$\boxed{\text{mass}}$ $\eta_t + H(u_x + v_y) = 0$

$\rightarrow \eta_{tt} + f^2\eta - gH(\eta_{xx} + \eta_{yy}) = 0$

solutions have the form $\eta = \mathrm{Re}\{\hat{\eta} \exp \overbrace{i(kx + ly - \omega t)}^{\text{"phase" } \varphi \text{ "phi"}}\}$

with dispersion relation $\omega = (K^2 c_0^2 + f^2)^{\frac{1}{2}}$ (*), $K^2 = k^2 + l^2$

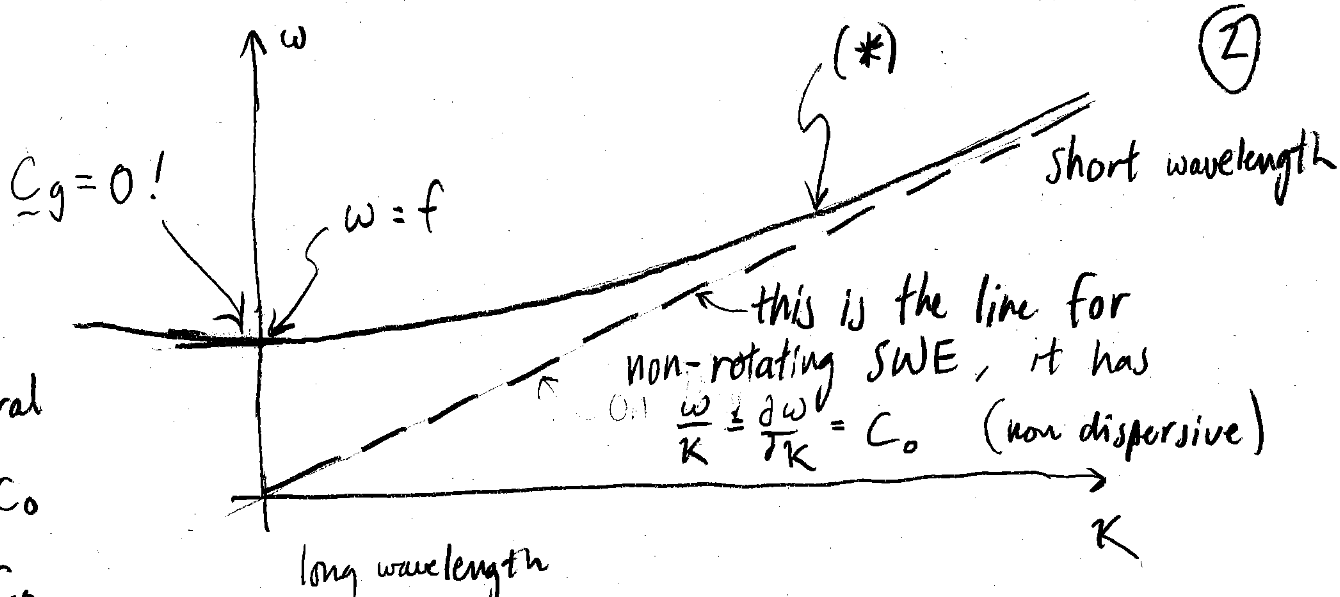
$c_0 = \sqrt{gH}$



recall phase speed $c_p = \frac{\omega}{K}$ in direction $\perp$ to wave crests

energy moves at "group velocity" $\underset{\sim}{c}_g \equiv \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}\right)$

Poincaré waves are dispersive $|c_p| \neq |\underset{\sim}{c}_g|$



In general
$C_p > C_0$
$C_g < C_0$
for Poincaré Waves

To understand the physics, consider the case where $l = 0 \Rightarrow \frac{\partial}{\partial y} = 0$, $K = k$ $\quad ||||\Rightarrow c_p$

Then $\boxed{x \text{ mom}}$ $\quad u_t - fv = -g\eta_x$

$\boxed{y \text{ mom}}$ $\quad v_t + fu = 0$

$\boxed{\text{mass}}$ $\quad \eta_t + H u_x = 0$

And assume $\hat{\eta} = \eta_0$ (real)
$\eta = \eta_0 \cos(kx - \omega t) = \eta_0 \cos\varphi$
waves move in positive x-direction for k positive

What are u + v?

from $\boxed{\text{mass}}$ $\quad u = \frac{-1}{H} \int \eta_t \, dx + \text{const.}$ — 0 assume no mean flow

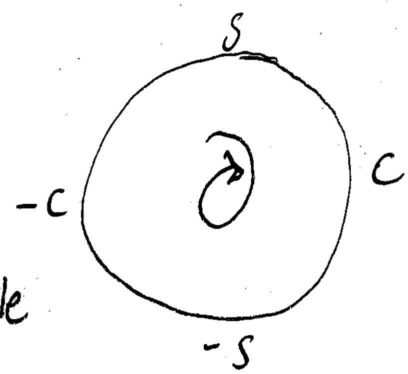
$$= -\frac{\eta_0}{H}(-\omega)\frac{1}{k}\cos\varphi = \boxed{\frac{\eta_0}{H}\frac{\omega}{k}\cos\varphi = u}$$

then from $\boxed{y \text{ mom}}$

$$v = -f \int u \, dt + \cancelto{0}{\text{const.}}$$

$$= -f \, \frac{\eta_0}{H} \, \frac{\cancel{\omega}}{k} \left(\frac{1}{-\cancel{\omega}} \right) \sin\varphi = \boxed{\frac{\eta_0}{H} \, \frac{f}{k} \sin\varphi = v}$$

¿ Helpful trick to remember derivatives and integrals of sine (s) & cosine (c)



Derivative moves clockwise around circle
Integral " counter-clockwise " "

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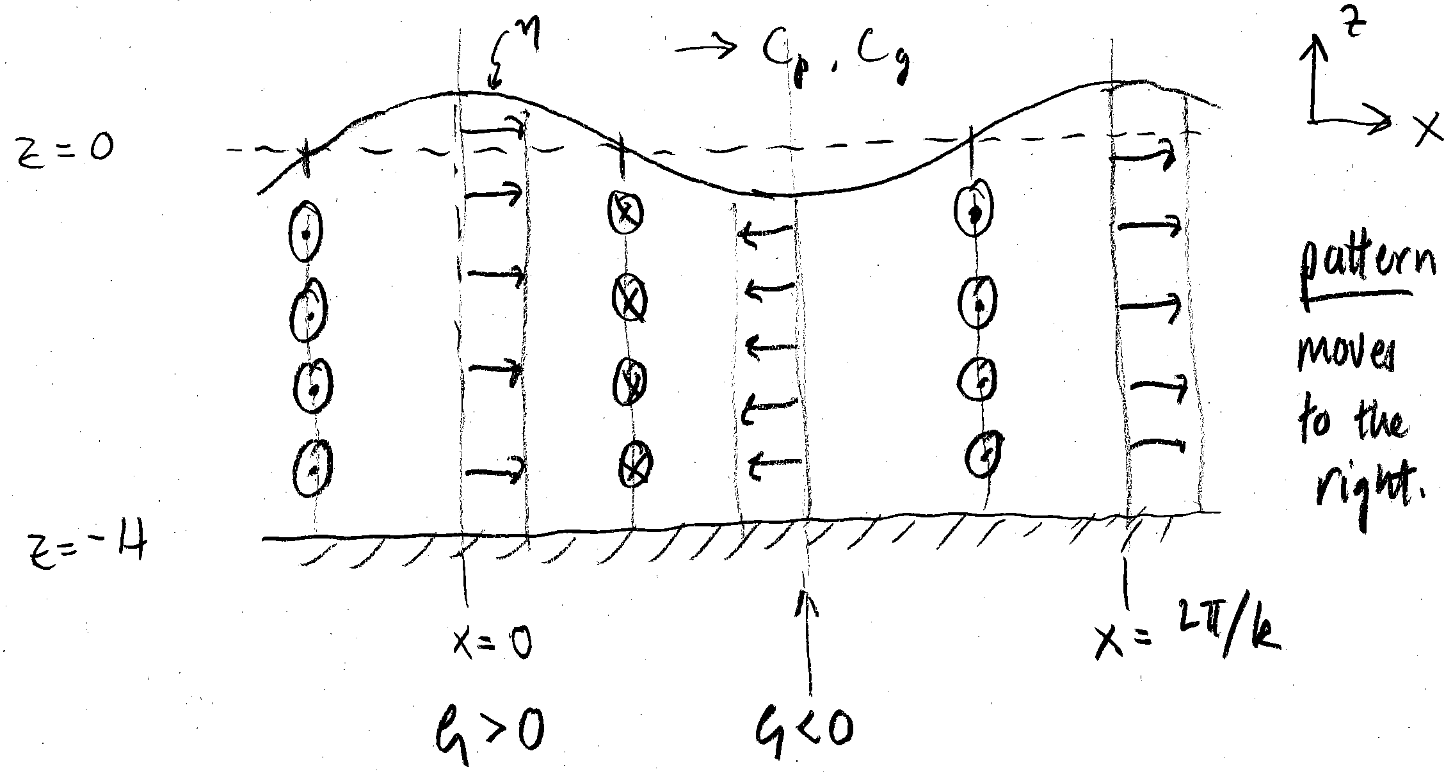
also note that $\int u \, dt = \Delta x$ so $\boxed{y \text{ mom}} \Rightarrow \Delta v = -f \Delta x$

☆ In the absence of other forces Coriolis turns small displacements into large velocities!

Graphically the solution looks like



Note $\zeta = v_x - \cancel{u_y}$ and it is easy to verify that $\zeta = \frac{f}{H}\eta$

<u>In the limit of long wavelength</u> $k, l \to 0$ and $\omega \to f$

so $u = \frac{\eta_0}{H}\frac{f}{k}\cos(-ft) = \mathcal{U}\cos ft$

$v = \frac{\eta_0}{H}\frac{f}{k}\sin(-ft) = -\mathcal{U}\sin ft$

assume $\eta_0 \to 0$ as $k \to 0$ and define $\frac{\eta_0}{H}\frac{f}{k} = \mathcal{U}$

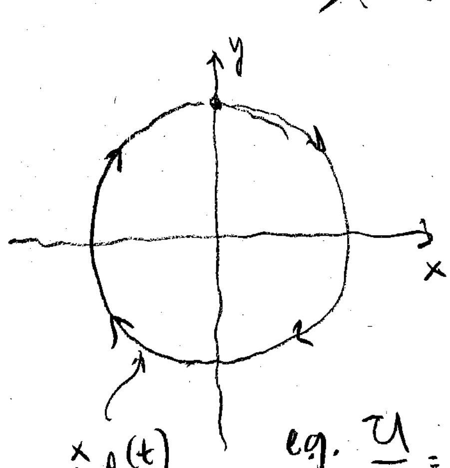
$\Rightarrow$ (no surface height and no spatial structure!)

Fluid parcel trajectory:

$x_p = \int u\,dt = \frac{\mathcal{U}}{f}\sin ft$

$y_p = \int v\,dt = \frac{\mathcal{U}}{f}\cos ft$

is a <u>circle</u>



Called "Inertial Oscillations"

Radius $= \frac{\mathcal{U}}{f}$

eg. $\frac{\mathcal{U}}{f} = \frac{0.5\ \mathrm{m\,s^{-1}}}{10^{-4}\ \mathrm{s^{-1}}} = 5\ \mathrm{km}$