$\boxed{3.2}$

$\boxed{\text{Poincaré Waves}}$

Recall "advective scaling" $\frac{1}{T} = \frac{U}{L}$

However many important motions are time-dependent with <u>wave-like solutions</u> which have constant T (period), and L (wavelength) even as $U \to 0$

$$\therefore \quad \frac{1}{T} \text{ may be } \gg \frac{U}{L}$$

so we retain term 1 in the scaling of $\boxed{\text{mass}}$

leaving

$$\boxed{\text{mass}} \quad \overset{1}{\eta_t} + \overset{2}{H(u_x + v_y)} = 0$$

$$\frac{\varepsilon}{T} \qquad \frac{HU}{L}$$

$$\Rightarrow \frac{1}{2} = \frac{\varepsilon}{T}\frac{L}{HU} = \underbrace{\left(\frac{\varepsilon}{H}\right)}_{\ll 1}\underbrace{\left(\frac{L}{UT}\right)}_{\gg 1} \sim \mathcal{O}(1\,?)$$

we may also linearize $\boxed{x, y \text{ mom}}$ when $\frac{1}{T} \gg \frac{U}{L}$

e.g. $$\frac{Du}{Dt} = u_t + \underbrace{u u_x + v u_y}_{} + w \cancel{u_z} \simeq u_t$$

scales $\quad \frac{U}{T} \quad \frac{U^2}{L}$

$\quad (1) \quad (2)$

no vertical shear for $\frac{H}{L} \ll 1$, $\rho = \text{const.}$

$$\Rightarrow \frac{(2)}{(1)} = \frac{TU}{L} \ll 1 \text{ to drop } (2)$$

Thus our linearized equations are

$\boxed{x \text{ mom}}$ $\quad u_t - fv = -g\eta_x \quad \Rightarrow u_{xt} = f v_x - g\eta_{xx}$

$\boxed{y \text{ mom}}$ $\quad v_t + fu = -g\eta_y \quad \Rightarrow v_{yt} = -f u_y - g\eta_{yy}$

$\boxed{\text{mass}}$ $\quad \eta_t + H(u_x + v_y) = 0$

$$\frac{\partial}{\partial t}\boxed{\text{mass}} \Rightarrow \eta_{tt} + H(u_{xt} + v_{yt}) = 0$$

$$\therefore \eta_{tt} - gH(\eta_{xx} + \eta_{yy}) + \underbrace{Hf(v_x - u_y)}_{} = 0 \quad (+)$$

or $Hf\zeta$ ("zeta")
where $\zeta = v_x - u_y$
vertical component of vorticity

★ for $f=0$ this is a wave equation with wave speed $c_0 = \sqrt{gH}$

For $f \neq 0$ we need two more steps to find ζ in terms of η:

Form: $\frac{\partial}{\partial x}\boxed{y\text{ mom}} - \frac{\partial}{\partial y}\boxed{x\text{ mom}}$

$$\Rightarrow \underbrace{(v_x - u_y)_t}_{\zeta} + f\underbrace{(u_x + v_y)}_{= -\frac{\eta_t}{H} \text{ from } \boxed{\text{mass}}} = -g\overbrace{(\cancel{\eta_{xy}} - \cancel{\eta_{xy}})}^{0} = 0$$

$$\Rightarrow \quad H\zeta_t = f\eta_t$$

so, if ζ and $\eta = 0$ initially, then $H\zeta = f\eta$

substituting this into $(+)$

$$\Rightarrow \quad \boxed{\eta_{tt} + f^2\eta - gH(\eta_{xx} + \eta_{yy}) = 0} \quad (++)$$

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for plane waves on an infinite f-plane we seek solutions of the form

$$\eta = \underset{\text{"real part of"}}{Re}\Big\{ \underset{\text{possibly complex}}{\hat{\eta}} \exp i(kx + ly - \omega t) \Big\}$$

recall $e^{i\theta} = \cos\theta + i\sin\theta$

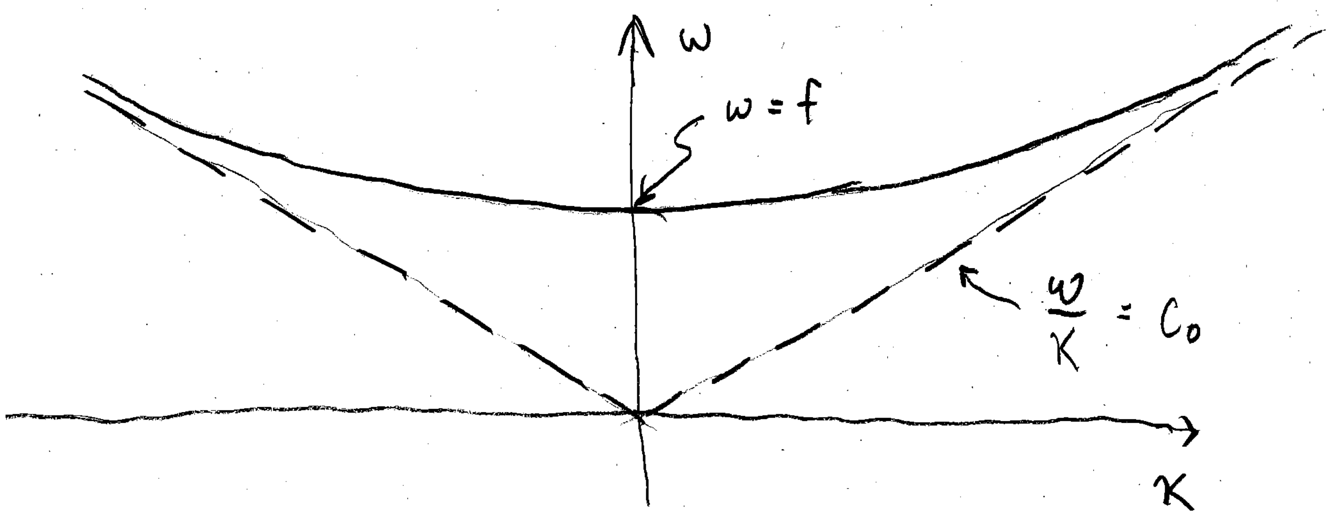
subitituting into (++)

$$\Rightarrow \quad -\omega^2 + f^2 + gH(k^2 + l^2) = 0$$

$\Rightarrow$ "dispersion relation" (assume $\omega > 0$)

$$\omega = \left(K^2 C_0^2 + f^2 \right)^{\frac{1}{2}}$$

$K^2 = k^2 + l^2$, $C_0 = \sqrt{gH}$



called "Poincaré Waves"