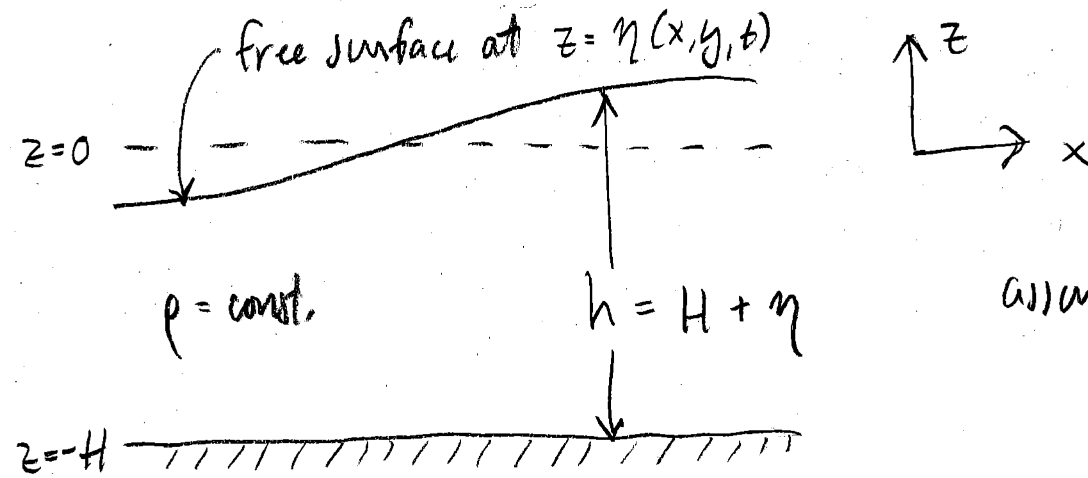
The Shallow Water Equations (SWE)

- a useful simplification to start study waves with rotation and potential vorticity -

Consider flow of a single, homogeneous layer on an f-plane:



assume $\frac{H}{L} \ll 1$

$\boxed{x \text{ mom}}$ $\frac{Du}{Dt} - fv = -\frac{1}{\rho} p_x$

$\boxed{y \text{ mom}}$ $\frac{Dv}{Dt} + fu = -\frac{1}{\rho} p_y$

$\boxed{z \text{ mom}}$ $p_z = -\rho g$ hydrostatic because $H/L \ll 1$

$\boxed{\text{mass}}$ $\nabla \cdot \underset{\sim}{u} = 0$ incompressible because ρ = const.

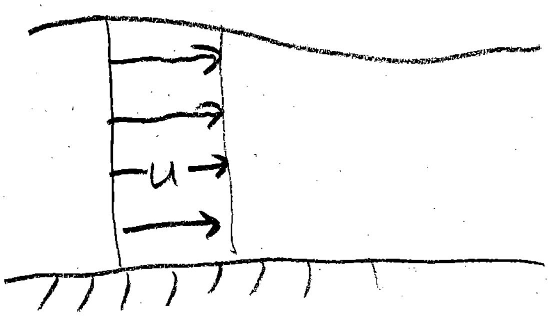
Taking $\frac{\partial}{\partial z}\boxed{x \text{ mom}}$ — ignore nonlinear terms

$$\Rightarrow \quad \frac{\partial u_z}{\partial t} - f v_z = -\frac{1}{\rho} p_{xz} = +\cancel{\frac{g \rho_x}{\rho}} = 0$$

(because ρ = const.)

$\therefore$ if u_z and $v_z = 0$ initially, then $u_z = 0$ for all time. From $\frac{\partial}{\partial z}\boxed{y \text{ mom}}$ we find $v_z = 0$ also.

☆ No vertical shear, a substantial simplification!



Then recall from lecture $\boxed{2.1}$ that we may write the pressure gradients in terms of η: $-\frac{1}{\rho} p_x = -g \eta_x$

$$\Rightarrow \quad \boxed{\begin{array}{ll} \boxed{x \text{ mom}} & \frac{Du}{Dt} - f v = -g \eta_x \\ \boxed{y \text{ mom}} & \frac{Dv}{Dt} + f u = -g \eta_y \end{array}}$$

Need an equation for η

note $\frac{Dv}{Dt} = v_t + u v_x + v v_y + \underbrace{\cancel{w u_z}}_{0}$, same for $\frac{Du}{Dt}$

from $\boxed{\text{mass}}$ $\nabla \cdot \underset{\sim}{u} = 0$

$$\int_{-H}^{\eta} \left[u_x + v_y + w_z = 0 \right] dz$$

$$\Rightarrow \quad \underbrace{w|_{\eta}}_{} - \underbrace{w|_{-H}}_{uH_x + vH_y} + hu_x + hv_y = 0$$

recall $u_z = v_z = 0$

$$\left\{ \begin{array}{l} uH_x + vH_y \\ = \dfrac{D\eta}{Dt} = \eta_t + u\eta_x + v\eta_y \end{array} \right\} = \eta_t + u(\eta + H)_x + v(\eta + H)_y$$

$$= \eta_t + uh_x + vh_y$$

$\therefore$ $\boxed{\text{mass}}$ $\boxed{\eta_t + (hu)_x + (hv)_y = 0}$ works with topography $H(x,y)$

* so we have $\boxed{\text{3 equations}}$ in u, v, η ✓

All are non linear. For small perturbations away from rest we may linearize them. Assume H = const.

consider $\boxed{\text{mass}}$, and scale $[\eta] = \varepsilon$ assuming $\varepsilon \ll H$

$\boxed{\text{Mass}}$
$$\eta_t + H(u_x + v_y) + \eta(u_x + v_y) + \eta_x u + \eta_y v = 0$$

scales
$$\frac{\varepsilon}{T} \qquad \frac{HU}{L} \qquad \frac{\varepsilon U}{L} \qquad \frac{\varepsilon U}{L} \qquad \frac{\varepsilon U}{L} = 0$$

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$$

term 2 $\gg$ 3, 4, + 5 so the main balance is

$$\boxed{\eta_t + Hu_x + Hv_y}$$

But <u>wait a minute</u> ... earlier we assumed "advective scaling" where $\frac{1}{T} \sim \frac{U}{L}$ implying we would also drop 1 vs. 2, leaving "horizontally nondivergent flow"

$$u_x + v_y = 0$$

This is satisfied by purely geostrophic flow (u_g, v_g) defined by $-fv_g = -g\eta_x \Rightarrow v_g = \frac{g}{f}\eta_x$

similarly $\quad u_g = -\frac{g}{f}\eta_y$

since $\quad u_{gx} + v_{gx} = \frac{g}{f}(-\eta_{xy} + \eta_{xy}) = 0$ ✓

Note: had we allowed $H(x,y)$ then for steady geostrophic flow $(Hu_g)_x + (Hv_g)_y = 0$

$$\Rightarrow H(\underbrace{\nabla_H \cdot \underset{\sim}{u}_g}_{=0 \text{ as before}}) + \underbrace{\underset{\sim}{u}_g \cdot \nabla_H H}_{\Rightarrow \text{ flow is along isobaths}} = 0$$