2.5 Rotating, Stratified Flow on a Sphere, the f-plane

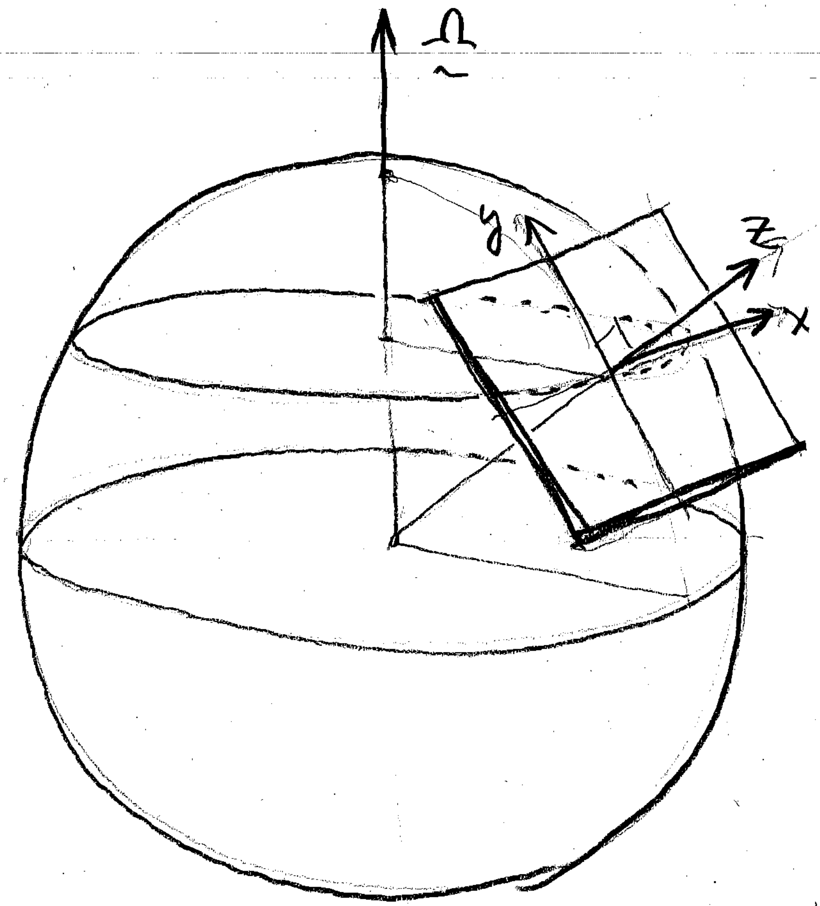
Our equations – now <u>assumed</u> to be hydrostatic and Boussinesq $\left(\frac{H}{L} \ll 1 \text{ and } \frac{\rho'}{\rho_0} \ll 1 \right)$ are:

$\boxed{\underset{\sim}{x} \text{ mom}}$ $$\frac{D\underset{\sim}{u}}{Dt} + 2\underset{\sim}{\Omega} \times \underset{\sim}{u} = -\frac{1}{\rho_0} \nabla p - \frac{g\rho}{\rho_0}\hat{k}$$

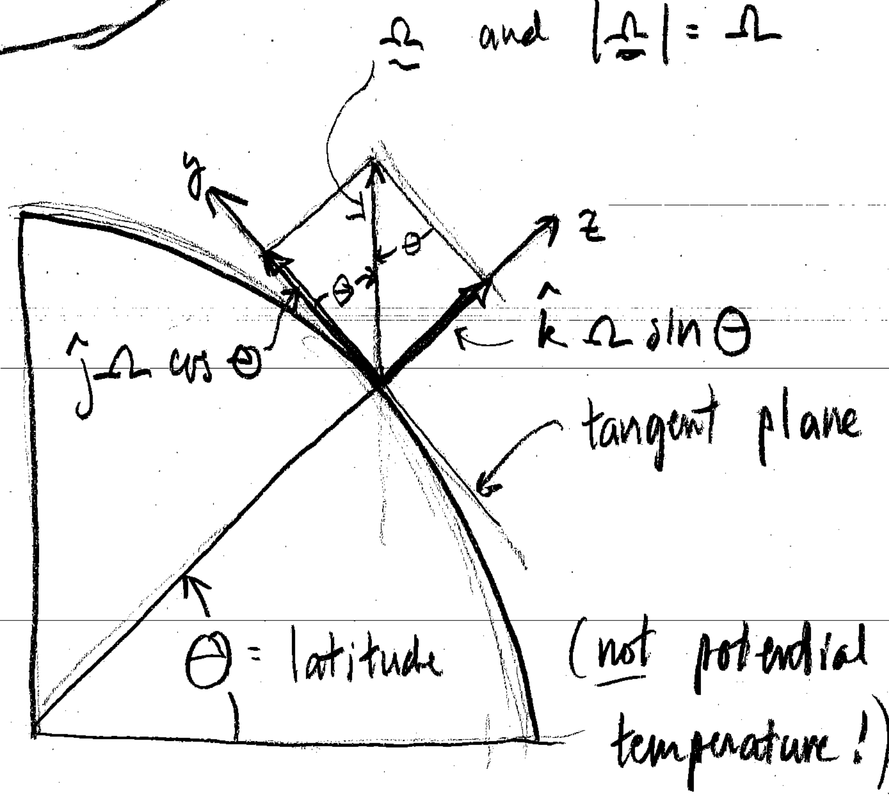
ρ_0 = const. background (average) density

$\boxed{\text{mass}}$ $$\nabla \cdot \underset{\sim}{u} = 0$$

Then, since the atm. or ocean is a thin layer, and many motions of interest are not affected by the sphericity of Earth, consider the motion to be taking place on a <u>plane</u>, locally tangent to Earth surface:



And we can write $\underset{\sim}{\Omega}$ in terms of components in the x-, y-, and z-directions (with associated unit vectors ($\hat{\imath}, \hat{\jmath}, \hat{k}$))



$$\underset{\sim}{\Omega} = \left(\hat{\imath}\,\overset{\text{zero}}{\emptyset},\ \hat{\jmath}\,\Omega\cos\theta,\ \hat{k}\,\Omega\sin\theta\right)$$

Then

$$2\underset{\sim}{\Omega} \times \underset{\sim}{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2\Omega\cos\theta & 2\Omega\sin\theta \\ u & v & w \end{vmatrix} = \hat{i}\left(2\Omega\cos\theta\, w - 2\Omega\sin\theta\, v\right) + \hat{j}\left(2\Omega\sin\theta\, u\right) - \hat{k}\left(2\Omega\cos\theta\, u\right)$$

So our momentum equations are

$$\frac{Du}{Dt} + (\underset{①}{2\Omega\cos\theta\, w} - \underset{②}{2\Omega\sin\theta\, v}) = -\frac{1}{\rho_0} p_x$$

$$\frac{Dv}{Dt} + 2\Omega\sin\theta\, u = -\frac{1}{\rho_0} p_y$$

$$\frac{Dw}{Dt} - 2\Omega\cos\theta\, u = -\frac{1}{\rho_0} p_z - \frac{g\rho}{\rho_0}$$

Since $\frac{H}{L} \ll 1$ and $W = U\frac{H}{L}$ we may neglect ① vs. ②. In the homework you will show that $\boxed{z \text{ mom}}$ is still dominantly hydrostatic.

So we have

$\boxed{\text{x mom}}$ $\quad \frac{Du}{Dt} - 2\Omega \sin\theta\, v = -\frac{1}{\rho_0} p_x$

$\boxed{\text{y mom}}$ $\quad \frac{Dv}{Dt} + 2\Omega \sin\theta\, u = -\frac{1}{\rho_0} p_y$

$\boxed{\text{z mom}}$ $\quad p_z = -\rho g$

$\boxed{\text{mass}}$ $\quad \nabla \cdot \underset{\sim}{u} = 0$

Then defining $\boxed{f \equiv 2\Omega \sin\theta}$ the Coriolis frequency

$2\Omega = 1.45 \times 10^{-4}\ s^{-1}$

and $\boxed{f \cong 10^{-4}\ s^{-1}}$ at mid-latitudes

~~May be written more compactly as~~

$\boxed{\underset{\sim}{x} \text{ mom}}$ $\quad \frac{D\underset{\sim}{u}}{Dt} + f\hat{k} \times \underset{\sim}{u} = -\frac{1}{\rho_0}\nabla p - \frac{g\rho}{\rho_0}\hat{k}$ (includes negligible Dw/Dt term)

- If the motion has limited latitude range (tens of degrees or 1000's of km) then we may approximate f by a constant $f_0 = 2\Omega \sin\Theta_0$ where Θ_0 is the central latitude of the motion.
- This is called the "f-plane"