2.4

$\boxed{\text{The Hydrostatic Approximation}}$ also $\boxed{\text{Boussinesq Approximation}}$

a scaling argument

Q: how good is $\frac{\partial p}{\partial z} = -\rho g$?

- we neglect rotation – it will be in the homework

Equations

$\boxed{x,y \text{ mom}}$ $\rho \frac{D \underset{\sim}{u}_H}{Dt} = -\nabla_H p$ $\underset{\sim}{u}_H = (u, v)$

$\nabla_H = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$\boxed{z \text{ mom}}$ $\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \rho g$

$\boxed{\text{mass}}$ $\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla_H \cdot \underset{\sim}{u}_H + \frac{\partial w}{\partial z} = 0$

Let [ ] denote "scale of"

and let $\rho = \bar{\rho}(z) + \rho'(\underset{\sim}{x}, t)$

$p = \bar{p}(z) + p'(\underset{\sim}{x}, t)$

Observed velocity scale $[\underset{\sim}{u}_H] = \mathcal{U}$

Observed horizontal length scale $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = \frac{1}{L}$

" vertical " " $[\partial/\partial z] = \frac{1}{H}$

☆ For most important atm/ocn flows $H \ll L$

Define $[w] = W$ unknown so far

Observed $[\rho, \bar{\rho}] = \rho_{00}$, and $[\rho'] = \rho_1$

☆ Assume that $\rho_1 \ll \rho_{00}$ (when is this <u>not</u> a good idea?)

☆ Assume "advective time scale": $\left[\frac{\partial}{\partial t}\right] = \frac{1}{L/\mathcal{U}} = \frac{\mathcal{U}}{L}$

Scaling $\boxed{\text{mass}}$ $\left[\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \underset{\sim}{u} = 0\right]$

$$\Rightarrow \underbrace{\left(\frac{1}{\rho}\rho_t + \frac{1}{\rho}(u\rho_x + v\rho_y)\right.}_{\frac{\rho_1}{\rho_{00}}\frac{\mathcal{U}}{L}} + \underbrace{\frac{1}{\rho}w\rho_z}_{\frac{\rho_1}{\rho_{00}}\frac{W}{H}} + \underbrace{(u_x + v_y)}_{\frac{\mathcal{U}}{L}} + \underbrace{w_z}_{\frac{W}{H}} = 0$$

$$\Rightarrow \quad \frac{\rho_1}{\rho_{00}}\frac{\mathcal{U}}{L} + \frac{\rho_1}{\rho_{00}}\frac{W}{H} \quad \frac{\mathcal{U}}{L} \quad \frac{W}{H} = 0$$

(1) (2) (3) (4)

(1) $\ll$ (3) & (2) $\ll$ (4) because $\rho_1/\rho_{00} \ll 1$

$\therefore \frac{\mathcal{U}}{L} = \frac{W}{H} \Rightarrow \boxed{W = \mathcal{U}\frac{H}{L}}$ and therefore $\left[\frac{D}{Dt}\right] = \frac{\mathcal{U}}{L}$

Then, scaling $\boxed{x, y \text{ mom}}$

$$\left[\rho \frac{D\underset{\sim}{u}_H}{Dt} = -\nabla_H p \right]$$

$$\Rightarrow \quad \rho_{00} \frac{U^2}{L} = \frac{[p']}{L} \Rightarrow \boxed{[p'] = \rho_{00} U^2}$$

like the "dynamic pressure"

Next, scale $\boxed{z \text{ mom}}$ and we <u>define</u> $\bar{p}$ as being in hydrostatic balance with $\bar{\rho} \Rightarrow \bar{p}_z = -\bar{\rho} g$

$$\left[\rho \frac{Dw}{Dt} = -\cancel{\bar{p}_z} - p'_z - \cancel{\bar{\rho}} g - \rho' g \right]$$

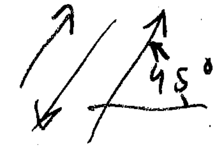
$$\rho_{00} \frac{WU}{L} = \frac{[p']}{H} + [\rho'] g$$

$$\text{or} \quad \underset{1}{\rho_{00} U^2 \frac{H^2}{L^2}} \qquad \underset{2}{\rho_{00} U^2} \qquad \underset{3}{[\rho'] g H}$$

For $H \ll L \Rightarrow$ 1 $\ll$ 2, leaving only 3 to balance 2

$\therefore \quad p'_z = -\rho' g$, and the vertical acceleration is $\mathcal{O}\left(\frac{H^2}{L^2}\right) \times$ perturbation pressure gradient!

Summary: under conditions where $[\rho'] \ll [\rho]$, and $\left(\frac{H}{L}\right)^2 \ll 1$, hydrostatic balance is a good approximation, even for perturbation quantities.

Note: this is not true for "steep" internal waves 

Another important consequence of $[\rho'] \ll [\rho]$ is the $\boxed{\text{Boussinesq Approximation}}$

defining $\rho = \rho_0 + \rho'(\underline{x}, t)$ with $\rho_0 = \text{constant}$ and $[\rho'] \ll \rho_0$

From our scaling of $\boxed{\text{mass}}$ we found $\left[\frac{1}{\rho}\frac{D\rho}{Dt}\right] \ll \left[\nabla \cdot \underset{\sim}{u}\right]$

☆☆ this applies to the individual terms of $\nabla \cdot \underset{\sim}{u}$, not their sum!

So we have

$\boxed{\text{mass}}$ $\nabla \cdot \underset{\sim}{u} = 0$ (*) approximately "incompressible"

and for $\boxed{\underline{x}\text{ mom}}$ $\rho \frac{D\underset{\sim}{u}}{Dt}$ is dominated by $\rho_0 \frac{D\underset{\sim}{u}}{Dt}$, so we may write

Keep full ρ so we can have $\rho' \propto \rho'_z$

$\boxed{\underline{x}\text{ mom}}$ $$\frac{D\underset{\sim}{u}}{Dt} = -\frac{1}{\rho_0}\nabla p - \frac{\rho g}{\rho_0}\hat{k} \quad (**)$$

(*) + (**) are the Boussinesq forms – doesn't require $\frac{H}{L} \ll 1$